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Variability of Measures of Weapons Effectiveness

Volume VII: Effectiveness Indices in Stick Delivery of Weapons

B O Sivazlian

UNIVERSITY OF FLORIDA
DEPARTMENT OF INDUSTRIAL AND SYSTEMS ENGINEERING
GAINESVILLE, FLORIDA 32611

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The probability of kill of a fragment sensitive target due to the stick delivery of multiple weapons subject to ballistic and aiming error is computed. It is assumed that the probability of kill associated with each weapon can be approximated by the Carleton damage function. Both the ballistic errors and the aiming errors are assumed to have a Gaussian distribution in each of the range and deflection directions. Each weapon is subject to ballistic errors which are statistically independent, but the entire stick pattern is subject as a whole to aiming error. A decomposition principle is used to obtain a general expression for the probability of kill. The variance of the probability of kill is estimated assuming that the uncertainty is present in the measurements of the input parameters.

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Effectiveness Indices in Stick Delivery of Weapons

PREFACE

This report describes work done during the summer of 1984 by Dr B. D. Sivazlian, principal investigator, from the Department of Industrial and Systems Engineering, the University of Florida, Gainesville, Florida 32611, under Contract No. F08635-83-C-0202 with the Air Force Armament Laboratory (AFATL), Armament Division, Eglin Air Force Base, Florida 32542. The program manager was Mr Daniel A. McInnis (DLYW).

This work addresses itself to the problem of determining the probability of kill of a fragment sensitive target due to the stick delivery of multiple weapons subject to ballistic and aiming errors. It is assumed that the probability of kill associated with each weapon can be approximated by the Carleton damage function. Both the ballistic errors and the aiming errors are assumed to have a Gaussian distribution in each of the range and deflection directions.

The author has benefited from helpful discussions with Mr Jerry Bass, Mr Daniel McInnis and Mr Charles Reynolds who have contributed to the report through their comments.

FOR THE COMMANDER

Milton D. Kingcaid

MILTON D. KINGCAID, Colonel, USAF
Chief, Analysis and Strategic Defense Division

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TABLE OF CONTENTS

TABLE OF CONTENTS (CONCLUDED)

SECTION I

INTRODUCTION

In this report, the problem of determining the probability of kill of a fragment sensitive target when attacked by multiple weapons (all of them identical) is discussed. Both independent and stick delivery of weapons are considered. This class of problem was first considered by R. Snow and M. Ryan [2] in 1970. The methodology used in this report is closely related to this previous work. This is true, in particular, when it is assumed that for the stick delivery of weapons each weapon is subject to ballistic errors which are statistically independent, but the entire stick pattern is subject as a whole to aiming error. A decomposition principle which may be verified for single weapon delivery and independent delivery is used to obtain a general expression for the probability of kill.

For a stick or ripple delivery of n general purpose (GP) bombs, the probability of kill of a point target is affected by:

a. The damage function of the individual weapons.

b. The overlap between the weapons.

c. The stick delivery pattern.

d. The uncertainties due to the individual ballistic errors of each weapon.

e. The aiming error of the center of the stick pattern.

With a preset timing of the intervalometer, each weapon i, $i=1,2,\ldots,n$, is targeted or aimed at a mean point of impact MPI_i on the assumption that the center of the stick pattern is aimed at the point target. Although this center is usually taken to be the center of gravity of the MPI_i's, different

criteria would define different centers. The MPI_i's of all the weapons form the stick pattern.

The i^{th} weapon is assumed to be subject to an individual ballistic error about its MPI_i. This ballistic error is measured as the abscissa (range) and ordinate (deflection) distances between MPI_i and the actual point of impact of the weapon. These distances are assumed to be independently and Gaussian distributed with mean defined by the coordinates of the MPI_i and with known standard deviations. The ballistic errors of all n weapons are assumed to be independently distributed.

Assume the x-axis to be in the direction of range and the y-axis to be in the direction of deflection. The center of the stick pattern is assumed to be aimed at the point target located at (u,v). Let $(\bar{x},\bar{y})$ be the actual point of impact of the center of the stick pattern. This center is assumed to be subject to an aiming error. The aiming error is defined by the distances $\bar{x}$-u and $\bar{y}$-v and these are assumed to have a Gaussian distribution with mean zeroes and given standard deviations. The aiming errors in each of the x and y directions are assumed to be independent. Further, the aiming errors are assumed to be independent of any of the ballistic errors associated with the individual weapons.

To determine the probability of kill of the point target located at (u,v), one has to construct the damage function of the stick pattern through the inclusion of the ballistic errors. This is followed by setting up the expression for the probability of kill of the point target by incorporating the aiming error of the center of the stick pattern. Note that the relative coordinate distances between the MPI_i's, i=1,2,...,n, and the center location are invariant. It is also clear that if the center is subject to aiming error, the stick pattern as a whole is subject to aiming error. However, the

aiming errors at each MPI_i are dependent, and this dependency must be captured when determining the final probability of kill.

Before discussing the general problem of stick delivey of multiple weapons subject to ballistic and aiming errors, the following problems are dealt with in sequence:

- Single weapon delivery with ballistic error (no aiming error);
- Multiple weapons delivery with ballistic error (no aiming error);
- Multiple weapons delivery in a stick pattern: no ballistic error, aiming error present at the center;
- Single weapon delivery with ballistic and aiming errors.

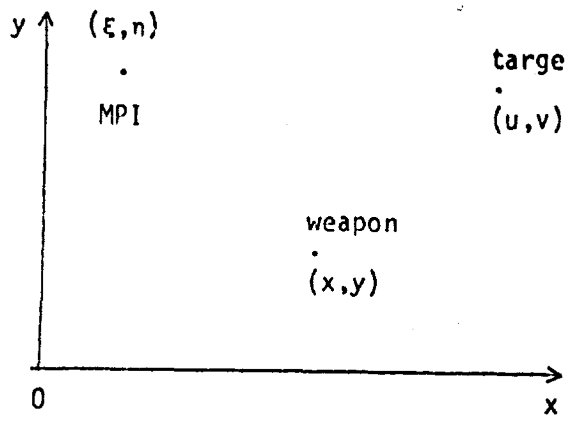
There are two reasons for investigating these special situations. First, they provide a means for checking the validity of the general results for multiple weapons by considering the single weapon as a special case. Second, a decomposition principle which can be verified for single weapon and independent delivery is later used to tackle the stick delivery problem.

Finally, the stick delivery of two weapons is discussed in detail. The problems discussed consist in the following:

- Derivation of the explicit expression for the probability of kill of a point target.
- Determination of the optimum stick pattern which maximizes the probability of kill.
- Estimation of the variance in the probability of kill given uncertainty in measurements in selected input parameters.

SECTION II

SINGLE WEAPON DELIVERY WITH BALLISTIC ERROR (NO AIMING ERROR)



Consider a single weapon and assume that the x-axis is taken in the direction of range and the y-axis is taken in the direction of deflection. Define as (u,v) the coordinates of the point target. Suppose that the weapon is aimed at (ξ,η) so that (ξ,η) is the MPI. Let the weapon subject to ballistic error impact at (x,y). The ballistic error in the range direction is $(x-\xi)$ and in the deflection direction is $(y-\eta)$. $(x-\xi)$ and $(y-\eta)$ are assumed to be independently distributed, each having a Gaussian distribution with zero mean and respective standard deviations σ_1 and σ_2. Thus, if $f_1(\cdot)$ and $f_2(\cdot)$ represent the respective probability density functions of $(x-\xi)$ and $(y-\eta)$ one has

$$f_1(x-\xi) = \frac{1}{\sigma_1 \sqrt{2\pi}} \exp\left[-\frac{(x-\xi)^2}{2\sigma_1^2}\right] \tag{1}$$

$$f_2(y-\eta) = \frac{1}{\sigma_2 \sqrt{2\pi}} \exp\left[-\frac{(y-\eta)^2}{2\sigma_2^2}\right] \tag{2}$$

Now the probability of kill at (u,v) given that the weapon impacts at (x,y) is assumed to be given by the three-parameter Carleton damage function

$$D(u-x,\ v-y) = D_0 \exp[\ -\ D_0[(\frac{u-x}{R_x})^2 + (\frac{v-y}{R_y})^2]] \tag{3}$$

Then:

Probability of kill at (u,v) =

$$P_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\text{Probability of kill at } (u,v)|\text{weapon impacts at } (x,y)]$$

$$[\text{Probability weapon impacts between } (x,y) \text{ and } (x+dx,y+dy)]$$

$$P_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-x,v-y)\ f_1(x-\xi)\ f_2(y-\eta)\ dx\ dy \tag{4}$$

Let $w = x-\xi$ and $z = y-\eta$, then

$$P_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D[(u-\xi)-w,\ (v-\eta)-z]\ f_1(w)\ f_2(z)\ dw\ dz \tag{5}$$

Using (1), (2) and (3) one obtains

$$P_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_0 \exp[-D_0[(\frac{(u-\xi)-w}{R_x})^2 + (\frac{(v-\eta)-z}{R_y})^2]]$$

$$\cdot \frac{1}{2\pi\sigma_1\sigma_2} \exp[-\frac{1}{2}(\frac{w^2}{\sigma_1^2} + \frac{z^2}{\sigma_2^2})]\ dw\ dz \tag{6}$$

This double integral can be explicitly evaluated to yield

$P_k = P_k\ (u-\xi,\ v-\eta)$

$$= \frac{R_x R_y}{2q_1 q_2} \exp\{-\frac{1}{2}[(\frac{u-\xi}{q_1})^2 + (\frac{v-\eta}{q_2})^2]\} \tag{7}$$

where

$$q_1^2 = \frac{R_x^2}{2D_0} + \sigma_1^2 \tag{8}$$

$$q_2^2 = \frac{R_y^2}{2D_0} + \sigma_2^2 \tag{9}$$

SECTION III

MULTIPLE WEAPONS DELIVERY WITH BALLISTIC ERROR (NO AIMING ERROR)

Suppose that n identical weapons are delivered. For the ith weapon, $i=1,2,\ldots,n$, let (ξ_i, η_i) be the point at which it is aimed or its MPI. The target is again assumed to be located at (u,v). Then, from (7), the probability of kill of the target due to weapon i is

$$P_{k_i}(u-\xi_i,\, v-\eta_i)$$

$$= \frac{R_x R_y}{2\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\left(\frac{u-\xi_i}{\sigma_1}\right)^2 + \left(\frac{v-\eta_i}{\sigma_2}\right)^2\right]\right\} \tag{10}$$

The probability of kill of the target due to all n weapons, assuming the weapons act independently and σ_1 and σ_2 are the same for all weapons, can be shown to be (see Appendix A).

$$\hat{P}_k^{(n)}(u,v) = 1 - \prod_{i=1}^{n}\left[1 - P_k(u-\xi_i,\, v-\eta_i)\right]$$

$$= 1 - \prod_{i=1}^{n}\left[1 - \frac{R_x R_y}{2\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\left(\frac{u-\xi_i}{\sigma_1}\right)^2 + \left(\frac{v-\eta_i}{\sigma_2}\right)^2\right]\right\}\right] \tag{11}$$

Note that this expression depends on (u,v) the position of the target, and (ξ_i,η_i) , $i=1,2,\ldots,n$, which is the mean point of impact of the ith weapon. (ξ_i,η_i) is not the actual point of impact of the ith weapon. Thus (11) is not actually a damage function which usually provides an expression for the probability of kill of a target as a function of the location of the target and the location of the actual point of impact of a weapon. Relation (11) holds true whether the n weapons are dropped independently or as a stick pattern, as long as the ballistic errors are assumed to be independent.

SECTION IV

THE CENTER OF THE STICK PATTERN

For a stick or ripple delivery, the arrays of all the MPI's defined by (ξ_i, η_i) for the ith weapon, $i=1,2,\ldots,n$, form the stick pattern.

The center of the stick pattern is usually used as the reference point to deliver the stick. For example, the center may be the point that would be aimed at a point target.

1. The Center of Gravity

This center is often taken to be the center of gravity $(\bar{\xi}, \bar{\eta})$ of the pattern. Thus, one has

$$\bar{\xi} = \frac{\sum_{i=1}^{n} \xi_i}{n} \quad \text{and} \quad \bar{\eta} = \frac{\sum_{i=1}^{n} \eta_i}{n} \qquad (12)$$

If (a_i, b_i) are the coordinates of the MPI of the ith weapon referred to $(\bar{\xi}, \bar{\eta})$, then

$$\sum_{i=1}^{n} a_i = 0 = \sum_{i=1}^{n} b_i$$

2. Other Measures of the Center

It is conceivable to select other centers of the stick pattern based on the formulation of specific criteria. One particular center which stands out is the one that maximizes the probability of kill of the point target.

Assume that there is no aiming error; then the problem under consideration consists in determining the optimum location of the point target (u,v)

(considered the aimpoint of the center) relative to the stick pattern so as to maximize the kill. Using expression (10), the problem can be stated as

$$\max_{u,v} \quad 1 - \prod_{i=1}^{n} \left[1 - \frac{R_x R_y}{2 q_1 q_2} \exp\left\{ -\frac{1}{2}\left[\left(\frac{u-\xi_i}{q_1}\right)^2 + \left(\frac{v-\eta_i}{q_2}\right)^2 \right]\right\} \right] \qquad (13)$$

SECTION V

MULTIPLE WEAPONS DELIVERY IN A STICK PATTERN; NO BALLISTIC ERROR, AIMING ERROR PRESENT AT THE CENTER

1. The Model

Suppose that n weapons are delivered in a stick mode. Let

(ξ_i, η_i) = coordinates of the MPI of the ith weapon, $i=1,2,\ldots,n$;

(x_i, y_i) = coordinates of the actual point of impact of the ith weapon;

$(\bar{\xi}, \bar{\eta})$ = coordinates of the MPI of the center of the stick pattern

(u,v) = coordinates of the point target;

$(\bar{x}, \bar{y})$ = coordinates of the actual point of impact of the center of the stick pattern.

Suppose now that the center is aimed at the target and is subject to aiming error. Thus, the MPI of the center $(\bar{\xi}, \bar{\eta})$ coincides with the point target (u,v). The aiming errors in the x and y directions are, respectively, $(\bar{x}-u)$ and $(\bar{y}-v)$. These are independently distributed, each having a Gaussian distribution with zero mean and each having respective standard deviations σ_x and σ_y. Let $g_1(\cdot)$ and $g_2(\cdot)$ be the respective probability density functions of $(\bar{x}-u)$ and $(\bar{y}-v)$; then

$$g_1(\bar{x}-u) = \frac{1}{\sqrt{2\pi}\ \sigma_x} \exp\left[-\frac{(\bar{x}-u)^2}{2\sigma_x^2}\right] \tag{14}$$

$$g_2(\bar{y}-v) = \frac{1}{\sqrt{2\pi}\ \sigma_y} \exp\left[-\frac{(\bar{y}-v)^2}{2\sigma_y^2}\right] \tag{15}$$

Clearly, each of the points (ξ_i, η_i), $i=1,2,\ldots,n$, will be subject to the same aiming error. Recall that the MPI for weapon i is (ξ_i, η_i). Let (x_i, y_i) be the actual point of impact of weapon i. Note that (u,v) is the center of

all the MPI_i points (ξ_i, η_i), whereas $(\bar{x}, \bar{y})$ is the center of all actual impact points (x_i, y_i). The entire stick pattern is assumed to simply be shifted under the influence of the aiming error.

Let (a_i, b_i) be the coordinates of the MPI of the ith weapon referred to (u,v), that is referred to the MPI of the center of the stick pattern. Then (a_i, b_i) will also be the coordinates of the actual impact point (x_i, y_i) of the ith weapon referred to $(\bar{x}, \bar{y})$ and one has the relations

$$\xi_i = u + a_i \;; \qquad \eta_i = v + b_i, \qquad i=1,2,\ldots,n \tag{16}$$

$$x_i = \bar{x} + a_i \;; \qquad y_i = \bar{y} + b_i, \qquad i=1,2,\ldots,n \tag{17}$$

The probability of kill at (u,v) due to all n weapons given that (x_i, y_i) is the point of impact of the ith weapon is, using (3):

$$\hat{P}_k = 1 - \prod_{i=1}^{n} \left[1 - D(u-x_i,\ v-y_i)\right] \tag{18}$$

Substituting (17) in (18) yields

$$\hat{P}_k = 1 - \prod_{i=1}^{n} \left[1 - D(u-\bar{x}-a_i,\ v-\bar{y}-b_i)\right] \tag{19}$$

To determine the net probability of kill P_{kT} of the target located at (u,v), one notes that

P_{kT} = probability of kill at (u,v) =

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}$ [Probability of kill at (u,v) | center of stick pattern impacts at $(\bar{x}, \bar{y})$]

[Probability that center of stick pattern impacts between $(\bar{x},\bar{y})$ and $(\bar{x} + d\bar{x},\ \bar{y} + d\bar{y})$]

Using (14), (15), and (19), one obtains:

$$P_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - D(u-\bar{x}-a_i,\ v-\bar{y}-b_i)]\}\ g_1(\bar{x}-u)\ g_2(\bar{y}-v)d\bar{x}\ d\bar{y} \qquad (20)$$

Making the changes in variables

$$w = \bar{x}-u\ ; \qquad z = \bar{y}-v$$

results in

$$P_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1- \prod_{i=1}^{n} [1 - D(-w-a_i,\ -z-b_i)\}\ g_1(w)\ g_2(z)\ dw\ dz \qquad (21)$$

which can be written as

$$P_{kT} = P_{kT}\ (a_1,\ a_2, \ldots,\ a_n;\ b_1, b_2, \ldots, b_n)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - D\ (w+a_i,\ z+b_i)]\}\ g_1(w)\ g_2(z)\ dw\ dz \qquad (22)$$

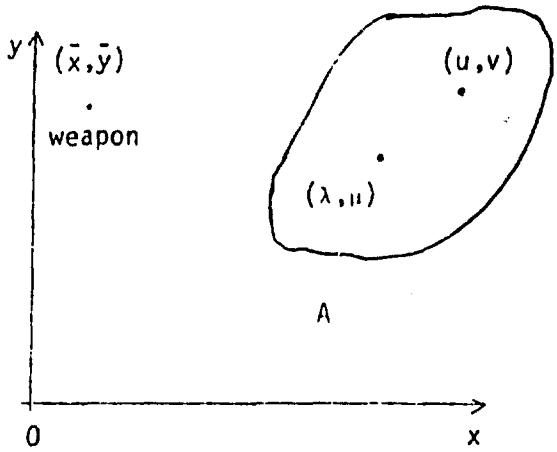
Using the explicit expressions for $D(\cdot,\cdot)$, $g_1(\cdot)$ and $g_2(\cdot)$ as given in (3), (14), and (15) yields

$$P_{kT}\ (a_1,\ a_2, \ldots, a_n;\ b_1, b_2, \ldots, b_n)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1- \prod_{i=1}^{n} [1 - D_0\ \exp[-D_0[(\frac{w+a_i}{R_x})^2 + (\frac{z+b_i}{R_y})^2]]]\}$$

$$\frac{1}{2\pi\sigma_x\sigma_y}\ \exp[-\frac{1}{2}(\frac{w^2}{\sigma_x^2} + \frac{z^2}{\sigma_y^2})]\ dw\ dz \qquad (23)$$

2. Finite Target Element



Consider now a finite target element of area A. Let (λ,μ) be the center of the target and (u,v) be a point on the target. It is required to determine the probability of kill at (u,v) when the center of the stick or ripple pattern is aimed at the center (λ,μ) of the target. Thus, (u,v) is no more the MPI of the center of the stick pattern.

Because of aiming error, the actual impact point of the center of the stick pattern is at $(\bar{x},\bar{y})$. It is assumed that $(\bar{x}-\lambda)$ and $(\bar{y}-\mu)$ are independently distributed each having the Gaussian probability density function

$$g_1(\bar{x}-\lambda) = \frac{1}{\sqrt{2\pi}\ \sigma_x} \exp\left[-\frac{(\bar{x}-\lambda)^2}{2\sigma_x^2}\right] \tag{24}$$

$$g_2(\bar{y}-\mu) = \frac{1}{\sqrt{2\pi}\ \sigma_y} \exp\left[-\frac{(\bar{y}-\mu)^2}{2\sigma_y^2}\right] \tag{25}$$

The probability of kill at (u,v) given that the center of the stick pattern has its impact point at $(\bar{x},\bar{y})$ is still given by expression (19) which is

$$\hat{P}_k = 1 - \prod_{i=1}^{n} [1 - D(u-\bar{x}-a_i,\ v-\bar{y}-b_i)] \tag{26}$$

where (a_i, b_i), $i=1,2,\ldots,n$, are the coordinates of the actual points of impact of the weapons referred to $(\bar{x},\bar{y})$ or equivalently, the coordinates of the MPI's of the weapons referred to $(\bar{\xi},\bar{\eta})$. The unconditional probability of kill at (u,v) is using (24), (25), and (26)

$$P_{kT}(u,v,\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - D(u-\bar{x}-a_i,\ v-\bar{y}-b_i)]\}$$

$$\cdot\ g_1(\bar{x}-\lambda)\ g_2(\bar{y}-\mu)\ d\bar{x}\ d\bar{y} \tag{27}$$

Let $w = \bar{x}-\lambda$ and $z = \bar{y}-\mu$ (28)

Expression (27) becomes

$$P_{kT}(u,v,\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - D(u-w-\lambda-a_i,\ v-z-\mu-b_i)]\ g_1(w)\ g_2(z)\ dw\ dz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - D_0 \exp[-D_0(\frac{u-w-\lambda-a_i}{R_x})^2 + (\frac{v-z-\mu-b_i}{R_y})^2\}]]\}$$

$$\frac{1}{2\pi\sigma_x\sigma_y} \exp\{-\frac{1}{2}(\frac{w^2}{\sigma_x^2} + \frac{z^2}{\sigma_y^2})\}\ dw\ dz \tag{29}$$

Select now arbitrarily the center of the weapon to be the origin of the system of coordinates; then $\lambda = 0 = \mu$. Then, the expression for the probability of kill at (u,v) written as $P_{kT}(u,v)$, as given by (29) becomes:

$$P_{kT}(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - D_0 \exp\{- D_0[(\frac{u-w-a_i}{R_x})^2 + (\frac{v-z-b_i}{R_y})^2]\}]\}$$

$$\frac{1}{2\pi\sigma_x\sigma_y} \exp\{- \frac{1}{2} (\frac{w^2}{\sigma_x^2} + \frac{z^2}{\sigma_y^2})\} \, dw \, dz \qquad (30)$$

Note that one recovers expression (21) by setting $u = \lambda$ and $v = \mu$ in expression (29), in which case the finite target element is reduced to a point.

Suppose that the target is a rectangle whose sides are parallel to the coordinate axes and have dimensions $2\ell_1$ and $2\ell_2$. The center of the rectangle coincides with the origin. The fractional coverage of the target is then

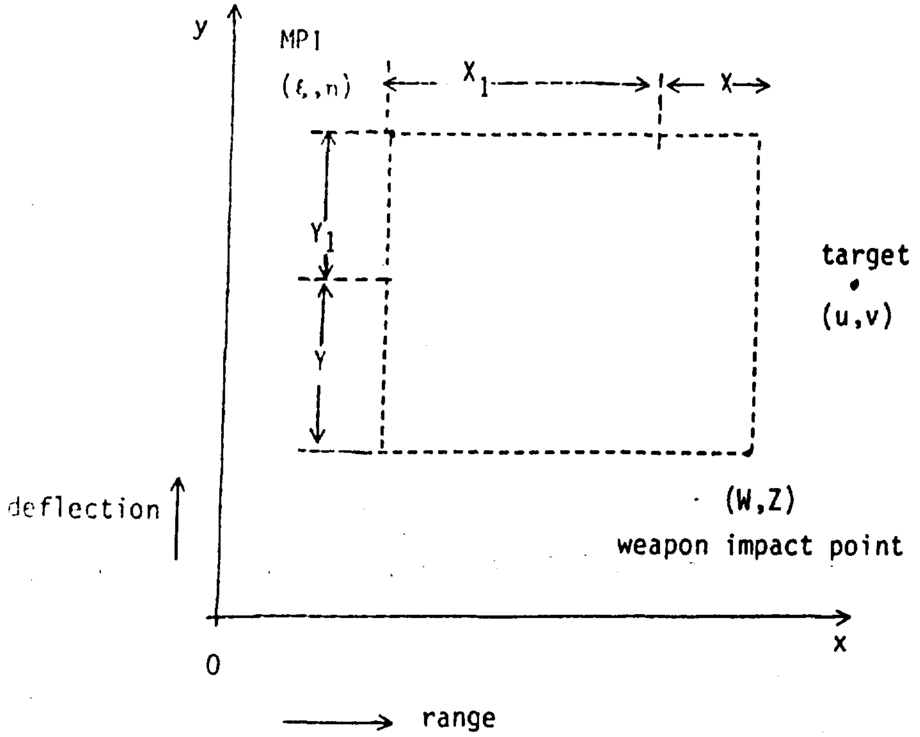
$$FC = \frac{1}{A} \int_A\int P_{kT}(u,v) \, du \, dv$$

$$= \frac{1}{4\ell_1\ell_2} \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} P_{kT}(u,v) \, du \, dv. \qquad (31)$$

SECTION VI

SINGLE WEAPON DELIVERY WITH BALLISTIC AND AIMING ERRORS

1. The Model



Consider a single weapon aimed at the point (ξ,η) and subject to both ballistic errors and aiming errors. The ballistic errors in the directions of range and deflection are, respectively, X_1 and Y_1. The random variables X_1 and Y_1 are assumed to be independent, each with zero mean and having the respective normal probability density functions:

$$f_1(x_1) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp[-\frac{x_1^2}{2\sigma_1^2}] \tag{32}$$

and

$$f_2(y_1) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp[-\frac{y_1^2}{2\sigma_y^2}] \tag{33}$$

The aiming errors in the directions of range and deflection are respectively X and Y. The random variables X and Y are assumed to be independent each with zero mean and having the respective normal probability density functions:

$$g_1(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp[-\frac{x^2}{2\sigma_x^2}] \tag{34}$$

and

$$g_2(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp[-\frac{y^2}{2\sigma_y^2}] \tag{35}$$

The random variables X_1, Y_1, X and Y are mutually independent. Let (W,Z) be the weapon impact point.

Clearly

$$\begin{aligned} W &= \xi + X_1 + X \\ Z &= \eta + Y_1 + Y \end{aligned} \tag{36}$$

The random variables W and Z are mutually independent and are normally distributed with respective means

$$E[W] = \xi \quad ; \quad E[Z] = \eta \tag{37}$$

and respective variances:

$$Var[W] = \sigma_1^2 + \sigma_x^2; \; Var[Z] = \sigma_2^2 + \sigma_y^2 \tag{38}$$

Let $h_W(w-\xi)$ and $h_Z(z-\eta)$ be the respective probability density functions of W and Z.

For a target located at (u,v), it is required to determine the probability of kill of the target assuming that the damage function is of the form (3) or

$$D(u-w,v-z) = D_0 \exp\{-D_0[(\frac{u-w}{R_x})^2 + (\frac{v-z}{R_y})^2]\} \tag{39}$$

Probability of kill at (u,v) =

$$\tilde{P}_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\text{Probability of kill at } (u,v) | \text{weapon impacts at } (w,z)]$$

$$[\text{Probability weapon impacts between } (w,z) \text{ and } (w+dw,\ z+dz)]$$

$$\tilde{P}_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-w,\ v-z)\ h_W(w-\xi)\ h_Z(z-\eta)\ dw\ dz \tag{40}$$

Let $\quad s = w-\xi \quad ; t = z-\eta$

$$\tilde{P}_k = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(u-s-\xi,\ v-t-\eta)\ h_W(s)\ h_Z(t)\ ds\ dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D[(u-\xi)-s,\ (v-\eta)-t]\ h_W(s)\ h_Z(t)\ ds\ dt \tag{41}$$

This integral is of the same form as (6); hence, by similarity to (7), its explicit value is

$$\tilde{P}_k = \tilde{P}_k(u-\xi,\ v-\eta)$$

$$= \frac{R_x R_y}{2Q_1 Q_2} \exp\{-\frac{1}{2} [(\frac{u-\xi}{Q_1})^2 + (\frac{v-\eta}{Q_2})^2]\} \tag{42}$$

where $$Q_1^2 = \frac{R_x^2}{2D_0} + \sigma_1^2 + \sigma_x^2 \tag{43}$$

$$Q_2^2 = \frac{R_y^2}{2D_0} + \sigma_2^2 + \sigma_y^2 \tag{44}$$

2. A Decomposition Principle

It is our purpose to show that the solution to the previous problem can be obtained in two stages, thus leading to a problem decomposition.

Stage 1 assumes only ballistic errors (no aiming errors) in which the damage function is of the form (3). The solution of this stage gives an expression for the probability of kill of a target point having form (7).

Stage 2 assumes only aiming errors (no ballistic errors) in which the damage function is of the form (7) obtained from the resulting solution of stage 1. It is shown that the solution to stage 2 gives expression (42) for the probability of kill of the target.

This problem decomposition is possible because:

a. Ballistic errors and aiming errors are independently distributed.

b. The Carleton damage function is similar in form to the Gaussian probability density function.

It is easy to verify that the solution approach is commutative; that is, in the problem decomposition, aiming error may be substitued to ballistic error in stage 1, whereas in stage 2 ballistic error may be substituted to aiming error.

We now proceed towards proving the decomposition principle and demonstrating the equivalence of the two approaches.

a. Stage 1

Suppose a single weapon is subject to ballistic errors and no aiming errors. Let

(ξ, η) = the MPI of the weapon;

(x,y) = the weapon impact point;

(u,v) = the location of the point target.

The ballistic error in the range direction is X_1 and in the deflection direction is Y_1. X_1 and Y_1 are independently distributed with zero means and with respective probability density functions given by (32) and (33). Thus,

$$x = X_1+\xi \quad \text{and} \quad y = Y_1+\eta$$

Assume now the damage function to be given by (3) or:

$$D(u-x,\ v-y) = D_0 \exp\{-D_0[(\frac{u-x}{R_x})^2 + (\frac{v-y}{R_y})^2]\} \tag{45}$$

Under these conditions, it was shown that the probability of kill of the target at (u,v) given that the weapon MPI is (ξ,η) is given by (7) or

$$P_k(u-\xi,\ v-\eta) = \frac{R_x R_y}{2q_1 q_2} \exp\{-\frac{1}{2}[(\frac{u-\xi}{q_1})^2 + (\frac{v-\eta}{q_2})^2]\} \tag{46}$$

where
$$q_1^2 = \frac{R_x^2}{2D_0} + \sigma_1^2 \tag{47}$$

$$q_2^2 = \frac{R_y^2}{2D_0} + \sigma_2^2 \tag{48}$$

b. Stage 2

Suppose a single weapon is subject to aiming error and no ballistic error. Let

(ξ,η) = the MPI of the weapon;

(x,y) = the weapon impact point;

(u,v) = the location of the point target.

For convenience, the same letter symbols are used in Stage 1, and this should not create any confusion.

The aiming error in the range direction is X and in the deflection direction is Y. X and Y are independently distributed with zero means and with respective probability density functions given by (34) and (35). Thus,

$$x = X + \xi \quad \text{and } y = Y + \eta$$

Assume now, that the damage function is given by (46) in which ξ is replaced by x and η is replaced by y. Thus, the new damage function is

$$P_k(u-x,v-y) = \frac{R_x R_y}{2q_1 q_2} \exp\left[-\frac{1}{2}\left[\left(\frac{u-x}{q_1}\right)^2 + \left(\frac{v-y}{q_2}\right)^2\right]\right] \tag{49}$$

where q_1 and q_2 are given by (47) and (48). Applying the usual conditional probability approach, it is easy to verify that the probability of kill at (u,v) when the MPI of the weapon is (ξ,η) is, using (34), (35), and (49):

$$\Pi_k = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_k(u-x,v-y)\, g_1(x-\xi)\, g_2(y-\eta)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_k[(u-\xi)-s,\ (v-\eta)-t]\, g_1(s)\, g_2(t)\, ds\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{R_x R_y}{2q_1 q_2} \exp\left[-\frac{1}{2}\left[\left(\frac{(u-\xi)-s}{q_1}\right)^2 + \left(\frac{(v-\eta)-t}{q_2}\right)^2\right]\right]$$

$$\cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{s^2}{\sigma_x^2} + \frac{t^2}{\sigma_y^2}\right)\right] ds\, dt \tag{50}$$

This integral is of the same form as (6) where, except for the multiplicative constant $\frac{R_x R_y}{q_1 q_2}$, the following substitutions have been made:

$D_0 \rightarrow \frac{1}{2}$, $R_x \rightarrow q_1$, $R_y \rightarrow q_2$, $\sigma_1 \rightarrow \sigma_x$, $\sigma_2 \rightarrow \sigma_y$. Thus from (7):

$$\Pi_k = \frac{R_x R_y}{q_1 q_2} \cdot \frac{q_1 q_2}{2 h_1 h_2} \exp\{-\frac{1}{2}[(\frac{u-\xi}{h_1})^2 + (\frac{v-\eta}{h_2})^2]\} \tag{51}$$

where using (8) and (9) one obtains

$$h_1^2 = q_1^2 + \sigma_x^2 \tag{52}$$

$$h_2^2 = q_2^2 + \sigma_y^2 \tag{53}$$

Substituting for the values of q_1^2 and q_2^2 yields

$$h_1^2 = \frac{R_x^2}{2D_0} + \sigma_1^2 + \sigma_x^2 \tag{54}$$

$$h_2^2 = \frac{R_y^2}{2D_0} + \sigma_2^2 + \sigma_y^2 \tag{55}$$

Comparing (54) to (43) and (55) to (44), one immediately notes that

$$h_1 = Q_1 \quad \text{and } h_2 = Q_2$$

Thus, expression (51) becomes

$$\Pi_k = \frac{R_x R_y}{2 Q_1 Q_2} \exp\{-\frac{1}{2}[(\frac{u-\xi}{Q_1})^2 + (\frac{v-\eta}{Q_2})^2]\} \tag{56}$$

Comparing (56) to (42), it follows that $\Pi_k = \tilde{P}_k$.

SECTION VII

MULTIPLE WEAPONS DELIVERY WITH BALLISTIC AND AIMING ERRORS

In the multiple delivery of weapons one has to consider two cases:

Case 1: Independent delivery of weapons. Here each weapon is subject to ballistic errors which are independently and identically distributed. In addition, each weapon is also subject to aiming errors which are independently and identically distributed. Finally, the ballistic errors are assumed to be independent of aiming errors.

Case 2: Stick or ripple delivery of weapons. Here each weapon is subject to ballistic errors which are independently and identically distributed. In addition, an aiming error is present on the entire stick pattern, and this error is usually associated with the center of the stick pattern. The stick pattern is assumed to act as a single rigid unit and thus respond as a whole integrated pattern to the presence of aiming error. It is obvious that in this case there is dependency in the aiming error of each weapon. The aiming error of the center of the stick pattern is assumed to be independent of the ballistic errors.

Each case is considered separately.

1. Independent Delivery of Weapons

Assume that there are n identical weapons. The ballistic errors on each weapon are assumed to be identically and independently normally distributed with means located at the same MPI of all the weapons and with standard deviation σ_1 in the x direction and σ_2 in the y direction. Similarly, the aiming errors on each weapon are assumed to be identically and

independently normally distributed with means located at the same MPI of all the weapons and with standard deviation σ_x in the x direction and σ_y in the y direction. The ballistic errors are assumed to be independent of the aiming errors. Let

(x_i, y_i) = the actual point of impact of the ith weapon, $i=1,2,\ldots,n$;

(ξ, η) = the coordinates of the MPI of the weapons,

(u, v) = the coordinates of the point target.

Assume the damage function to be of the form (3) for each weapon, i.e.,

$$D(u-x_i,\ v-y_i) = D_0 \exp[-D_0[(\frac{u-x_i}{R_x})^2 + (\frac{v-y_i}{R_y})^2]] \tag{57}$$

The actual point of impact of the ith weapon, i.e., (x_i, y_i), is the result of the combined effect of the ballistic error and aiming error. This combined effect is the net delivery error which is the sum of the ballistic error and aiming error in each of the x and y directions. Thus, the following results:

1. In the x direction, a net delivery error $\hat{X}_i$ which is normally distributed with mean located at ξ and variance

$$\hat{\sigma}_X^2 = \sigma_1^2 + \sigma_x^2 \tag{58}$$

The probability density function of $\hat{X}_i$ is

$$f_{\hat{X}_i}(x_i - \xi) = \frac{1}{\hat{\sigma}_X \sqrt{2\pi}} \exp[-\frac{1}{2}(\frac{x_i-\xi}{\hat{\sigma}_X})^2] \tag{59}$$

2. In the y-direction, a net delivery error $\hat{Y}_i$ which is normally distributed with mean located at η and variance

$$\hat{\sigma}_Y^2 = \sigma_2^2 + \sigma_y^2 \tag{60}$$

The probability density function of $\hat{Y}_i$ is

$$f_{\hat{Y}_i}(y_i-\eta) = \frac{1}{\hat{\sigma}_Y\sqrt{2\pi}} \exp[-\frac{1}{2}(\frac{y_i-\eta}{\hat{\sigma}_Y})^2] \tag{61}$$

The probability of kill at (u,v) =

$$\hat{P}_k = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2n} [\text{Probability of kill at } (u,v)|\text{weapon i impacts at } (x_i,y_i)]$$

$$[\text{Probability weapon i impacts between } (x_i,y_i) \text{ and } (x_i+dx_i,y_i+dy_i)].$$

$$= \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2n} \{1 - \prod_{i=1}^{n} [1 - D(u-x_i,\ v-y_i)]\} \prod_{i=1}^{n} f_{\hat{X}_i}(x_i-\xi)\ f_{\hat{Y}_i}(y_i-\eta)\ dx_i\ dy_i$$

Let

$$\begin{aligned} x_i - \xi &= s_i \\ y_i - \eta &= t_i \end{aligned} \tag{63}$$

Then,

$$\hat{P}_k = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{2n}, \{1 - \prod_{i=1}^{n} [1 - D((u-\xi)-s_i,(v-\eta)-t_i)]\}$$

$$\prod_{i=1}^{n} f_{\hat{X}_i}(s_i)\ f_{\hat{Y}_i}(t_i)\ ds_i\ dt_i \tag{64}$$

Using the same approach as given in Appendix A, it may be verified that

$$\hat{P}_k = 1 - [1 - \frac{R_x R_y}{2Q_1 Q_2} \exp[-\frac{1}{2}[(\frac{u-\xi}{Q_1})^2 + (\frac{v-\eta}{Q_2})^2]]]^n \tag{65}$$

It may also be verified that the decomposition principle applies here, and that, similar to the single weapon case, the problem may be approached in two stages. In the first stage only ballistic errors are considered with the damage function being

$$\{1 - \prod_{i=1}^{n} [1 - D(u-x_i, v-y_i)]\}$$

$$= \{1 - \prod_{i=1}^{n} [1 - D_0 \exp\{- D_0 [(\frac{u-x_i}{R_x})^2 + (\frac{v-y_i}{R_y})^2\}]\} \tag{66}$$

In the second stage only aiming errors are considered with the damage function being

$$\{1 - \prod_{i=1}^{n} [1 - \frac{R_x R_y}{2 q_1 q_2} \exp\{- \frac{1}{2} [(\frac{u-x_i}{q_1})^2 + (\frac{v-y_i}{q_2})^2]\}]\} \tag{67}$$

where

$$q_1^2 = \frac{R_x^2}{2 D_0} + \sigma_1^2 \tag{68}$$

$$q_2^2 = \frac{R_y^2}{2 D_0} + \sigma_2^2 \tag{69}$$

2. Stick Delivery of Weapons

The stick delivery of weapons is significantly more complex to model. The following assumptions are usually made:

a. Each weapon is subject to ballistic errors which are assumed to be normally distributed, independent of each other and independent in each of the x and y directions.

b. The center of the stick pattern is subject to aiming error which is assumed to be normally distributed, independent in each of the x and y directions and independent of the ballistic errors.

The simulanteous incorporation of these two types of errors into a single model for determining the probability of kill of the target results in an expression which cannot be manipulated analytically. This arises from the fact that there is dependency in the aiming errors of the weapons.

An approximation to the problem can be obtained using the decomposition principle, although this cannot be verified. The problem is tackled in two stages. In the first stage, the weapons are assumed to be subject only to ballistic errors (no aiming errors present). In the second stage, the center of the stick pattern is assumed to be subject only to aiming error (no ballistic errors).

Stage 1

Assume that there are n identical weapons. The ballistic errors on each weapon are assumed to be identically and independently normally distributed with means located at the different MPI's of the weapons and with standard deviations σ_1 in the x direction and σ_2 in the y direction. Let

(x_i, y_i) = coordinates of the actual point of impact of the ith weapon, $i=1,2,\ldots,n$;

(ξ_i, η_i) = coordinates of the MPI of the ith weapon, $i=1,2,\ldots,n$;

(u,v) = coordinates of the point target.

Assume the damage function of the ith weapon to be

$$D(u-x_i,\ v-y_i) = D_0 \exp\{- D_0 \left[\left(\frac{u-x_i}{R_x}\right)^2 + \left(\frac{v-y_i}{R_y}\right)^2\right]\} \qquad (70)$$

The damage function of all n weapons is

$$\{1 - \prod_{i=1}^{n} [1 - D(u-x_i, v-y_i)]\} \tag{71}$$

The probability of kill of the target is immediately given by (11) or:

$$\hat{P}_k^{(n)}(u,v) = 1 - \prod_{i=1}^{n} [1 - \frac{R_x R_y}{2q_1 q_2} \exp\{-\frac{1}{2}[(\frac{u-\xi_i}{q_1})^2 + (\frac{v-\eta_i}{q_2})^2]\}] \tag{72}$$

Let now

$(\bar{\xi},\bar{\eta})$ = the center of the MPI of all n weapons;

(a_i,b_i) = the coordinates of the MPI of the ith weapon referred to its center.

Thus, $\xi_i = \bar{\xi} + a_i$; $\eta_i = \bar{\eta} + b_i$ (73)

Substituting (73) in (72) yields

$$\hat{P}_k^{(n)}(u,v) = \hat{P}_k^{(n)}(u-\bar{\xi},v-\bar{\eta})$$

$$= \{1 - \prod_{i=1}^{n} [1 - \frac{R_x R_y}{2q_1 q_2} \exp\{-\frac{1}{2}[(\frac{u-\bar{\xi}-a_i}{q_1})^2 + (\frac{v-\bar{\eta}-b_i}{q_2})^2]\}]\} \tag{74}$$

It is evident that in the presence of ballistic errors only, if the MPI of the center of the stick pattern coincides with the location of the point target (u,v), then

$$\bar{\xi} = u \quad \text{and} \quad \bar{\eta} = v$$

so that (74) becomes independent of u and v resulting in

$$\hat{P}_k^{(n)} = \{1 - \prod_{i=1}^{n} [1- \frac{R_x R_y}{2q_1 q_2} \exp\{-\frac{1}{2}[(\frac{a_i}{q_1})^2 + (\frac{b_i}{q_2})^2]\}]\} \tag{75}$$

Stage 2

Assume that for the stick pattern as a whole the center is subject to aiming error. Let

$(\bar{\xi},\bar{\eta})$ = coordinates of the MPI of the center of the stick pattern;

$(\bar{x},\bar{y})$ = coordinates of the actual point of impact of the center of the stick pattern;

(u,v) = coordinates of the point target.

The aiming errors in the x and y directions are, respectively, $(\bar{x}-\bar{\xi})$ and $(\bar{y}-\bar{\eta})$. These are independently and normally distributed with zero mean and with respective standard deviations σ_x and σ_y. Let $g_1(.)$ and $g_2(.)$ be the respective probability density functions of $(\bar{x}-\bar{\xi})$ and $(\bar{y}-\bar{\eta})$; then

$$g_1(\bar{x}-\bar{\xi}) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[-\frac{(\bar{x}-\bar{\xi})^2}{2\sigma_x^2}\right] \tag{76}$$

$$g_2(\bar{y}-\bar{\eta}) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[-\frac{(\bar{y}-\bar{\eta})^2}{2\sigma_y^2}\right] \tag{77}$$

The damage function is now assumed to be given by expression (74) obtained from stage 1, in which $\bar{\xi}$ is replaced by $\bar{x}$ and $\bar{\eta}$ is replaced by $\bar{y}$. Thus, the new damage function is:

$$\hat{P}_k^{(n)}(u-\bar{x},\ v-\bar{y}) =$$

$$\{1 - \prod_{i=1}^{n} [1 - \frac{R_x R_y}{2q_1 q_2} \exp[-\frac{1}{2}[(\frac{u-\bar{x}-a_i}{q_1})^2 + (\frac{v-\bar{y}-b_i}{q_2})^2]]]\} \tag{78}$$

where q_1 and q_2 are given, respectively, by

$$q_1^2 = \frac{R_x^2}{2\,D_0} + \sigma_1^2 \tag{79}$$

$$q_2^2 = \frac{R_y^2}{2\,D_0} + \sigma_2^2 \tag{80}$$

The probability of kill at (u,v) using the new damage function is now computed using conditional probabilities.

$\hat{\Pi}_{kT}$ = Probability of kill at (u,v) =

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty}$ [Probability of kill at (u,v)| center of the stick pattern impacts at $(\bar{x},\bar{y})$] [Probability that the center of the stick pattern impacts between $(\bar{x},\bar{y})$ and $(\bar{x}+d\bar{x},\ \bar{y}+d\bar{y})$]

Using (76), (77), and (78) results in

$$\hat{\Pi}_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}_k^{(n)}(u-\bar{x},\ v-\bar{y})\ g_1(\bar{x}-\bar{\xi})\ g_2(\bar{y}-\bar{\eta})\ d\bar{x}\ d\bar{y} \tag{81}$$

Let $s = \bar{x} - \bar{\xi}\ ;\quad t = \bar{y} - \bar{\eta}$ (82)

Substituting (82) in (81) yields

$$\hat{\Pi}_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}_k^{(n)}[(u-\bar{\xi})-s;\ (v-\bar{\eta})-t]\ g_1(s)\ g_2(t)\ ds\ dt \tag{83}$$

Or written explicitly, (83) becomes:

$$\hat{\Pi}_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - \frac{R_x R_y}{2q_1 q_2} \exp[-\frac{1}{2}[(\frac{(u-\bar{\xi})-s-a_i}{q_1})^2 + (\frac{(v-\bar{\eta})-t-b_i}{q_2})^2]]]\}$$

$$\cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp[-(\frac{s^2}{2\sigma_x^2} + \frac{t^2}{2\sigma_y^2})] \, ds \, dt \tag{84}$$

3. Special Cases

Two special cases are considered. In the first special case, the MPI of the center of the stick pattern is assumed to coincide with the point target. In the second special case, one assumes that no ballistic errors are present and a previously obtained result is recovered.

a. Special Case 1:

Here one assumes that $(\bar{\xi},\bar{\eta})$ coincides with (u,v)

or $\bar{\xi}=u$ and $\bar{\eta}=v$

Expression (84) becomes independent of $\bar{\xi},\bar{\eta}$, u, and v, and one obtains:

$$\hat{\Pi}_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1- \frac{R_x R_y}{2q_1 q_2} \exp\{-\frac{1}{2}[\frac{(s+a_i)^2}{q_1^2} + \frac{(t+b_i)^2}{q_2^2}]\}]\}$$

$$\cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp[-(\frac{s^2}{2\sigma_x^2} + \frac{t^2}{2\sigma_y^2})] \, ds \, dt \tag{85}$$

b. Special Case 2

In the absence of ballistic errors, it follows that

$\sigma_1 = 0$ and $\sigma_2 = 0$

Expressions (79) and (80) for q_1 and q_2 become

$$q_1 = \frac{R_x}{\sqrt{2\ D_0}} \quad ; \qquad q_2 = \frac{R_y}{\sqrt{2\ D_0}} \tag{86}$$

Thus
$$D_0 = \frac{R_x R_y}{2 q_1 q_2} \tag{87}$$

Substituting (86) and (87) in (84) results in

$$\bar{H}_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - D_0 \exp\{- D_0 [(\frac{(u-\bar{\xi})-s-a_i}{R_x})^2 + (\frac{(v-\bar{\eta})-t-b_i}{R_y})^2]\}]\}$$

$$\cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp[-(\frac{s^2}{2\sigma_x^2} + \frac{t^2}{2\sigma_y^2})]\ ds\ dt \tag{88}$$

In the case when the MPI of the center coincides with the point target, $u=\bar{\xi}$ and $v=\bar{\eta}$, and expression (23) is recovered.

c. Remark

The general expression (84) provides a means for calculating the probability of kill of a target point located at (u,v) given that the center of the stick pattern has a MPI located at $(\bar{\xi},\bar{\eta})$. This expression is also a function of all the (a_i,b_i)'s, i=1,2,...,n, which are the coordinates of the MPI's of the weapons referred to the center $(\bar{\xi},\bar{\eta})$.

In case when the center is selected as the center of gravity of the MPI's of all n weapons, the quantities a_i and b_i must satisfy the following relations

$$\sum_{i=1}^{n} a_i = 0 = \sum_{i=1}^{n} b_i$$

As a general rule, expressions (84) and (85) can be evaluated in closed form. In particular, when n=1, $a_1 = 0 = b_1$ and expression (84) reduces to (50) with $\xi = \bar{\xi}$ and $\eta = \bar{\eta}$. The reduced closed form is given by (56).

4. Finite Target Element

Consider now a finite target element of area A. Let (λ,μ) be the center of the target and (u,v) be a point on the target. Suppose that the center of the stick pattern has its MPI at (λ,μ) so that

$$\bar{\xi} = \lambda \quad \text{and} \quad \bar{\eta} = \mu$$

It is required to determine the probability of kill at (u,v). The approach used here is different than the one used in Section V-2, although one obtains similar results.

Expression (84) becomes

$$\hat{\Pi}_{kT}(u,v,\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - \frac{R_x R_y}{2q_1q_2} \exp\{-\frac{1}{2}[(\frac{u-\lambda-s-a_i}{q_1})^2 + (\frac{v-\mu-t-b_i}{q_2})^2]\}]\} \cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp[-(\frac{s^2}{2\sigma_x^2} + \frac{t^2}{2\sigma_y^2})] \, ds\, dt \qquad (89)$$

Select now arbitrarily the center of the target to be the origin of the system of coordinates. Then $\lambda = 0 = \mu$ and the expression for the probability of kill at (u,v) written as $\Pi_{kT}(u,v)$, given by expression (89), becomes

$$\hat{\Pi}_{kT}(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{n} [1 - \frac{R_x R_y}{2q_1 q_2} \exp[-\frac{1}{2}[(\frac{u-s-a_i}{q_1})^2$$

$$+ (\frac{v-t-b_i}{q_2})^2]]]\} \cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp[-\frac{1}{2}(\frac{s^2}{\sigma_x^2} + \frac{t^2}{\sigma_y^2})]\, ds\, dt \qquad (90)$$

Note that this expression reduces to (30) when $\sigma_1 = 0 = \sigma_2$

Suppose that the target is a rectangle whose sides are parallel to the coordinate axis and have dimensions $2\ell_1$ and $2\ell_2$. The center of the rectangle coincides with the origin. The fractional coverage of the target is then

$$FC = \frac{1}{4\ell_1\ell_2} \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \hat{\Pi}_{kT}^{(n)}(u,v)\, du\, dv \qquad (91)$$

SECTION VIII

STICK DELIVERY OF TWO WEAPONS

1. Expression for the Probability of Kill

It shall be assumed that the center of the stick pattern $(\bar{\xi},\bar{\eta})$ is aimed at the point target located at $\bar{\xi} = u$ and $\bar{\eta} = v$. The expression for the probability of kill of the target is given by (85) with n=2. Rewriting the expression we obtain

$$\hat{\Pi}^{(2)}_{kT} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{2} [1 - \frac{R_x R_y}{2 q_1 q_2} \exp[-\frac{1}{2} [\frac{(s+a_i)^2}{q_1^2} + \frac{(t+b_i)^2}{q_2^2}]]]\}$$

$$\cdot \frac{1}{2\pi\sigma_x\sigma_y} \exp[-\frac{1}{2}(\frac{s^2}{\sigma_x^2} + \frac{t^2}{\sigma_y^2})] \, ds \, dt \qquad (92)$$

The following expressions are now defined:

$$U_i(s) = \frac{R_x}{\sqrt{2}\, q_1} \exp[-\frac{1}{2} \frac{(s+a_i)^2}{q_1^2}] \qquad i=1,2 \qquad (93)$$

$$V_i(s) = \frac{R_y}{\sqrt{2}\, q_2} \exp[-\frac{1}{2} \frac{(t+b_i)^2}{q_2^2}] \qquad i=1,2 \qquad (94)$$

$$g_1(s) = \frac{1}{\sqrt{2\pi}\, \sigma_x} \exp[-\frac{1}{2} (\frac{s^2}{\sigma_x^2})] \qquad (95)$$

$$g_2(t) = \frac{1}{\sqrt{2\pi}\, \sigma_y} \exp[-\frac{1}{2} (\frac{t^2}{\sigma_y^2})] \qquad (96)$$

Expression (92) may be written as follows:

$$\hat{\Pi}_{kT}^{(2)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{1 - \prod_{i=1}^{2} [1 - U_i(s)V_i(t)]\} \, g_1(s)g_2(t) \, ds \, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [U_1(s)V_1(t) + U_2(s)V_2(t) - U_1(s)V_1(t)U_2(s)V_2(t)] \quad g_1(s) \, g_2(t) \, ds \, dt$$

$$= \int_{-\infty}^{\infty} U_1(s) \, g_1(s) \, ds \cdot \int_{-\infty}^{\infty} V_1(t) \, g_2(t) \, dt$$

$$+ \int_{-\infty}^{\infty} U_2(s) \, g_1(s) \, ds \cdot \int_{-\infty}^{\infty} V_2(t) \, g_2(t) \, dt$$

$$- \int_{-\infty}^{\infty} U_1(s) \, U_2(s) \, g_1(s) \, ds \cdot \int_{-\infty}^{\infty} V_1(t) \, V_2(t) \, g_2(t) \, dt \tag{97}$$

Each of the integrals in (97) can be evaluated using the results of Appendix B. Consider each of these integrals separately.

$$\int_{-\infty}^{\infty} U_1(s) \, g_1(s) \, ds = \frac{R_x}{\sqrt{2}\, q_1} \cdot \frac{1}{\sqrt{2\pi}\, \sigma_x} \sqrt{\frac{2\pi}{\gamma_2}} \exp[-\frac{1}{2} (\delta_2 - \frac{\epsilon_2^2}{\gamma_2})] \tag{98}$$

where
$$\gamma_2 = \frac{1}{q_1^2} + \frac{1}{\sigma_x^2} \tag{99}$$

$$\delta_2 = \frac{a_1^2}{q_1^2} + \frac{0}{\sigma_x^2} = \frac{a_1^2}{q_1^2} \tag{100}$$

$$\epsilon_2 = \frac{a_1}{q_1^2} + \frac{0}{\sigma_x^2} = \frac{a_1}{q_1^2} \tag{101}$$

Substituting (99), (100) and (101) in (98) results in

$$\int_{-\infty}^{\infty} U_1(s)\ q_1(s)\ ds = \frac{R_x}{\sqrt{2}\ q_1} \cdot \frac{1}{\sqrt{2\pi}\ \sigma_x} \cdot \sqrt{\frac{2\pi}{\frac{1}{q_1^2} + \frac{1}{\sigma_x^2}}}\ \exp\{-\frac{1}{2}[\frac{a_1^2}{q_1^2} - \frac{\frac{a_1^2}{q_1^4}}{\frac{1}{q_1^2} + \frac{1}{\sigma_x^2}}]\}$$

$$= \frac{R_x}{\sqrt{2}}\ \frac{1}{\sqrt{q_1^2 + \sigma_x^2}}\ \exp[-\frac{1}{2}\frac{a_1^2}{q_1^2 + \sigma_x^2}] \qquad (102)$$

By analogy one has

$$\int_{-\infty}^{\infty} V_1(t)\ g_2(t)\ dt = \frac{R_y}{\sqrt{2}}\ \frac{1}{\sqrt{q_2^2 + \sigma_y^2}}\ \exp[-\frac{1}{2}\frac{b_1^2}{q_2^2 + \sigma_y^2}] \qquad (103)$$

$$\int_{-\infty}^{\infty} U_2(s)\ g_1(s)\ ds = \frac{R_x}{\sqrt{2}}\ \frac{1}{\sqrt{q_1^2 + \sigma_x^2}}\ \exp[-\frac{1}{2}\frac{a_2^2}{q_1^2 + \sigma_x^2}] \qquad (104)$$

$$\int_{-\infty}^{\infty} V_2(t)\ g_2(t)\ dt = \frac{R_y}{\sqrt{2}}\ \frac{1}{\sqrt{q_2^2 + \sigma_y^2}}\ \exp[-\frac{1}{2}\frac{b_2^2}{q_2^2 + \sigma_y^2}] \qquad (105)$$

It remains to compute the last two integrals in (97). Again using the results of Appendix B, one obtains:

$$\int_{-\infty}^{\infty} U_1(s)\ U_2(s)\ g_1(s)\ ds$$

$$= \frac{R_x}{\sqrt{2}\ q_1} \cdot \frac{R_x}{\sqrt{2}\ q_1} \cdot \frac{1}{\sqrt{2\pi}\ \sigma_x}\ \sqrt{\frac{2\pi}{\gamma_3}}\ \exp[-\frac{1}{2}(\delta_3 - \frac{\varepsilon_3^2}{\gamma_3})] \qquad (106)$$

where $\gamma_3 = \frac{1}{q_1^2} + \frac{1}{q_1^2} + \frac{1}{\sigma_x^2} = \frac{2}{q_1^2} + \frac{1}{\sigma_x^2}$ (107)

$$\delta_3 = \frac{a_1^2}{q_1^2} + \frac{a_2^2}{q_1^2} + \frac{0}{\sigma_x^2} = \frac{(a_1^2 + a_2^2)}{q_1^2} \tag{108}$$

$$\epsilon_3 = \frac{a_1}{q_1^2} + \frac{a_2}{q_1^2} + \frac{0}{\sigma_x^2} = \frac{(a_1 + a_2)}{q_1^2} \tag{109}$$

Substituting (107), (108), and (109) in (106) results in

$$\int_{-\infty}^{\infty} U_1(s)\, U_2(s)\, g_1(s)\, ds = \frac{R_x^2}{2q_1^2} \cdot \frac{1}{\sigma_x} \cdot \frac{1}{\sqrt{\frac{2}{q_1^2} + \frac{1}{\sigma_x^2}}}$$

$$\exp\{-\frac{1}{2}\, [\frac{(a_1^2 + a_2^2)}{q_1^2} - \frac{\frac{(a_1 + a_2)^2}{q_1^4}}{\frac{2}{q_1^2} + \frac{1}{\sigma_x^2}}]\}$$

$$= \frac{R_x^2}{2q_1 \sqrt{2\sigma_x^2 + q_1^2}} \exp\{-\frac{1}{2q_1^2}\, [(a_1^2 + a_2^2) - \frac{(a_1 + a_2)^2\, \sigma_x^2}{2\sigma_x^2 + q_1^2}]\} \tag{110}$$

By analogy one also has

$$\int_{-\infty}^{\infty} V_1(s)\, V_2(s)\, g_2(s)\, ds =$$

$$\frac{R_y^2}{2q_2 \sqrt{2\sigma_y^2 + q_2^2}} \exp\{-\frac{1}{2q_2^2}\, [(b_1^2 + b_2^2) - \frac{(b_1 + b_2)^2\, \sigma_y^2}{2\sigma_y^2 + q_2^2}]\} \tag{111}$$

The explicit expression for the target probability of kill may now be obtained by substituting (102), (103), (104), (105), (110), and (111) in (97) which results in the following:

$$
\begin{aligned}
\hat{\Pi}_{kT}^{(2)} = {} & \frac{R_x R_y}{2} \frac{1}{\sqrt{q_1^2 + \sigma_x^2}} \frac{1}{\sqrt{q_2^2 + \sigma_y^2}} \exp\left\{-\frac{1}{2}\left(\frac{a_1^2}{q_1^2 + \sigma_x^2} + \frac{b_1^2}{q_2^2 + \sigma_y^2}\right)\right\} \\
& + \frac{R_x R_y}{2} \frac{1}{\sqrt{q_1^2 + \sigma_x^2}} \cdot \frac{1}{\sqrt{q_2^2 + \sigma_y^2}} \exp\left\{-\frac{1}{2}\left(\frac{a_2^2}{q_1^2 + \sigma_x^2} + \frac{b_2^2}{q_2^2 + \sigma_y^2}\right)\right\} \\
& - \frac{R_x^2}{2q_1 \sqrt{2\sigma_x^2 + q_1^2}} \exp\left\{-\frac{1}{2q_1^2}\left[(a_1^2 + a_2^2) - \frac{(a_1 + a_2)^2 \sigma_x^2}{2\sigma_x^2 + q_1^2}\right]\right\} \\
& \cdot \frac{R_y^2}{2q_2 \sqrt{2\sigma_y^2 + q_2^2}} \exp\left\{-\frac{1}{2q_2^2}\left[(b_1^2 + b_2^2) - \frac{(b_1 + b_2)^2 \sigma_y^2}{2\sigma_y^2 + q_2^2}\right]\right\}
\end{aligned}
\tag{112}
$$

2. <u>Optimum Stick Pattern</u>

The first question that arises is whether it is possible to select the values of a_1, a_2, b_1, and b_2 in an optimum fashion so as to maximize the value of the probability of kill for the two-weapon system, as dictated by expression (112). Such values do indeed exist. Rather than solve the general problem two special cases will be considered corresponding to a drop pattern along the x-axis in the direction of range, or along the y-axis in the direction of deflection.

Let

$$C_1 = \frac{R_x R_y}{2} \frac{1}{\sqrt{q_1^2 + \sigma_x^2}} \cdot \frac{1}{\sqrt{q_2^2 + \sigma_y^2}} \tag{113}$$

$$C_2 = \frac{R_x^2}{2q_1\sqrt{2\sigma_x^2 + q_1^2}} \cdot \frac{R_y^2}{2q_2\sqrt{2\sigma_y^2 + q_2^2}} \tag{114}$$

Substituting (113) and (114) in (112) results in

$$\begin{aligned}
\hat{\Pi}_{kT}^{(2)} &= C_1 \exp\{-\frac{1}{2}(\frac{a_1^2}{q_1^2 + \sigma_x^2} + \frac{b_1^2}{q_2^2 + \sigma_y^2})\} \\
&+ C_1 \exp\{-\frac{1}{2}(\frac{a_2^2}{q_1^2 + \sigma_x^2} + \frac{b_2^2}{q_2^2 + \sigma_y^2})\} \\
&- C_2 \exp\{-\frac{1}{2q_1^2}[(a_1^2 + a_2^2) - \frac{(a_1 + a_2)^2\sigma_x^2}{2\sigma_x^2 + q_1^2}]\} \\
&\cdot \exp\{-\frac{1}{2q_2^2}[(b_1^2 + b_1^2) - \frac{(b_1 + b_2)^2\sigma_y^2}{2\sigma_y^2 + q_2^2}]\}
\end{aligned} \tag{115}$$

Without loss in generality, it shall be assumed that the MPI of the stick pattern coincides with the coordinates of the origin which is also the location of the point target.

Case 1: $a_1 = -a_2 = a;\ \ b_1 = 0 = b_2$

The stick delivery is along the x-axis, and the MPI of each of the two weapons is equidistant from the origin (the target point). Expression (115) as a function of a, the decision variable, becomes

$$\hat{\Pi}_{kT}^{(2)}(a) = 2\,C_1 \exp(-\frac{1}{2}\ \frac{a^2}{q_1^2 + \sigma_x^2}) - C_2 \exp(-\frac{a^2}{q_1^2}) \tag{116}$$

Differentiating (116) with respect to a and setting the result equal to zero,

one obtains

$$\frac{d\hat{\Pi}^{(2)}_{kT}(a)}{da} = 0 = a\left[\frac{C_1}{q_1^2 + \sigma_x^2} \exp\left(-\frac{1}{2}\frac{a^2}{q_1^2 + \sigma_x^2}\right) - \frac{C_2}{q_1^2} \exp\left(-\frac{a^2}{q_1^2}\right)\right] \qquad (117)$$

The solution to this equation yields the value $a = 0$, which corresponds to a minimum, and

$$\frac{C_2}{C_1} \frac{q_1^2 + \sigma_x^2}{q_1^2} = \exp\left[a^2 \left(\frac{1}{q_1^2} - \frac{1}{2}\frac{1}{q_1^2 + \sigma_x^2}\right)\right] \qquad (118)$$

or

$$a^2 = \frac{\ln\left(\frac{C_2}{C_1} \frac{q_1^2 + \sigma_x^2}{q_1^2}\right)}{\frac{1}{q_1^2} - \frac{1}{2}\frac{1}{q_1^2 + \sigma_x^2}} \qquad (119)$$

which corresponds to a maximum.

Note here that Case 1 corresponds in practice to a drop of two bombs from the middle of the aircraft at a distance 2a from each other with the target equidistant from the MPI's.

<u>Case 2</u>: $a_1 = 0 = a_2$; $b_1 = -b_2 = b$

The stick delivery is along the y-axis, and the MPI of each of the two weapons is equidistant from the origin (the target point). In practice, this will correspond to the simultaneous drop of two bombs from under the aircraft wings assuming each bomb is a distance b apart from the body of the aircraft.

Expression (115) as a function of b, the new decision variable, becomes:

$$\hat{\Pi}^{(2)}_{kT}(b) = 2\, C_1 \exp\left(-\frac{1}{2}\frac{b^2}{q_2^2 + \sigma_y^2}\right) - C_2 \exp\left(-\frac{b^2}{q_2^2}\right) \qquad (120)$$

Differentiating (120) with respect to b and setting the result equal to zero, one obtains:

$$\frac{d\hat{I}_{kT}^{(2)}(b)}{db} = 0 = b\left[\frac{C_1}{q_2^2 + \sigma_y^2} \exp\left(-\frac{1}{2}\frac{b^2}{q_2^2 + \sigma_y^2}\right) - \frac{C_2}{q_2^2} \exp\left(-\frac{b^2}{q_2^2}\right)\right] \tag{121}$$

The solution to this equation yields the value b = 0 which corresponds to a minimum, and

$$\frac{C_2}{C_1}\frac{q_2^2 + \sigma_y^2}{q_2^2} = \exp\left[b^2\left(\frac{1}{q_2^2} - \frac{1}{2}\frac{1}{q_2^2 + \sigma_y^2}\right)\right] \tag{122}$$

or

$$b^2 = \frac{\ln\left(\frac{C_2}{C_1}\frac{q_2^2 + \sigma_y^2}{q_2^2}\right)}{\frac{1}{q_2^2} - \frac{1}{2}\frac{1}{q_2^2 + \sigma_y^2}} \tag{123}$$

which corresponds to a maximum.

3. Example

The following data are given

$R_x = 15$ ft ; $R_y = 30$ ft ; $D_0 = 1.00$

$\sigma_1 = 30$ ft ; $\sigma_2 = 20$ ft ; $\sigma_x = 150$ ft ; $\sigma_y = 100$ ft

From (79)

$$q_1^2 = \frac{R_x^2}{2\,D_0} + \sigma_1^2$$

$$= \frac{(15)^2}{2} + (30)^2 = 1{,}012.5$$

Hence, $q_1 = 31.82$

From (80)

$$q_2^2 = \frac{R_y^2}{2\, n_0} + \sigma_2^2$$

$$= \frac{(30)^2}{2} + (20)^2 = 850$$

Hence, $q_2 = 29.15$

From (113)

$$C_1 = \frac{R_x R_y}{2} \quad \frac{1}{\sqrt{q_1^2 + \sigma_x^2}} \cdot \frac{1}{\sqrt{q_2^2 + \sigma_y^2}}$$

$$= \frac{(15)(30)}{2} \frac{1}{\sqrt{1,012.5 + (150)^2}} \quad \frac{1}{\sqrt{850 + (100)^2}}$$

$$= \frac{(15)(30)}{2} \quad \frac{1}{153.34} \cdot \frac{1}{104.16} = .014,087$$

From (114)

$$C_2 = \frac{R_x^2}{2q_1 \sqrt{2\sigma_x^2 + q_1^2}} \cdot \frac{R_y^2}{2q_2 \sqrt{2\sigma_y^2 + q_2^2}}$$

$$C_2 = \frac{(15)^2}{(2)(31.82) \sqrt{(2)(150)^2 + 1,012.5}} \cdot \frac{(30)^2}{(2)(29.15) \sqrt{(2)(100)^2 + 850}}$$

$$= (.016,482)(.106,911) = .001,762,1$$

Optimum value of a

From (119)

$$a^2 = \frac{\ln\left(\frac{C_2}{C_1} \frac{q_1^2 + \sigma_x^2}{q_1^2}\right)}{\frac{1}{q_1^2} - \frac{1}{2} \frac{1}{q_1^2 + \sigma_x^2}}$$

$$= \frac{\ln\left(\frac{.001,762,1}{.014,087} \cdot \frac{1,012.5 + (150)^2}{1,012.5}\right)}{\frac{1}{1,012.5} - \frac{1}{2} \frac{1}{1,012.5 + (150)^2}}$$

$$= \frac{\ln 2.904,8}{.000,966,389} = 1,103.45$$

Thus, $a = 33.22$ ft.

Optimum value of b

From (123)

$$b^2 = \frac{\ln\left(\frac{C_2}{C_1} \frac{q_2^2 + \sigma_y^2}{q_2^2}\right)}{\frac{1}{q_2^2} - \frac{1}{2} \frac{1}{q_2^2 + \sigma_y^2}}$$

$$= \frac{\ln\left(\frac{.001,762,1}{.014,087} \cdot \frac{850 + (100)^2}{850}\right)}{\frac{1}{850} - \frac{1}{2} \frac{1}{850 + (100)^2}}$$

$$= \frac{\ln (1.596,7)}{.001,130,4} = 413.96$$

$b = 20.34$ ft.

Computation of the Kill Probability

a. At the minimum value: a = 0 = b

From (116) or (120)

$$\hat{\Pi}_{kT}^{(2)} = 2C_1 - C_2$$

$$= (2)(.014{,}087) - (.001{,}766{,}21)$$

$$= .026{,}41.$$

b. At the optimum value: a = 33.22 ft, b=0

From (116)

$$\hat{\Pi}_{kT}^{(2)} = 2(.014{,}087)\ \exp\left[-\frac{1}{2}\frac{(33.22)^2}{1{,}012.5 + (150)^2}\right] - (.001{,}762{,}1)\ \exp\left[-\frac{(33.22)^2}{1{,}012.5}\right]$$

$$= (2)(.014{,}087(.976{,}81) - (.001{,}762{,}1)(.336{,}235)$$

$$= .026{,}93.$$

c. At the optimum value a = 0, b = 20.34 ft.

From (120)

$$\hat{\Pi}_{kT}^{(2)} = 2(.014{,}087)\ \exp\left[-\frac{1}{2}\frac{(20.34)^2}{850 + (100)^2}\right] - (.001{,}762{,}1)\ \exp\left[-\frac{(20.34)^2}{850}\right]$$

$$= (2)(.014{,}087)(.981{,}115) - (.001{,}762{,}1)(.614{,}636)$$

$$= .026{,}56.$$

At least for this example, an optimum configuration of a two bomb stick pattern does not seem to improve the probability of kill significantly.

4. Error Estimation in the Probability of Kill

Although it is theoretically possible to obtain a closed form expression for $\mathrm{Var}[\hat{\Pi}_{kT}^{(n)}]$ when uncertainty exists in the input parameters which are D_0, R_x, R_y, σ_1, σ_2, σ_x and σ_y, nevertheless, the resulting expression becomes extremely cumbersome and quite unmanageable even for the case of $n=2$ weapons. The methodology, however, is not different than the one developed in [3] where Taylor's series was used to obtain an approximate closed form expression for the variance of the probability of kill. The difficulty stems from three causes:

a. A total of seven input parameters may have inherent estimation errors which will, in general, be expressed by a 7x7 variance-covariance matrix, thus involving a total of 28 variance and covariance terms which must be specified numerically.

b. The general expression (which is (85)) involves n MPI's whose incorporation as general input variables adds to the problem complexity.

c. The form of the expression (85) involving terms under the product sign is not easily amenable to differentiation with respect to the seven input parameters. These operations must be performed in order to obtain the terms of the Taylor's series expansion.

For these reasons, the problem of variance estimation has been reduced to one consisting of the following:

a. The number of weapons involved is limited to 2($n=2$).

b. The stick pattern is the one for which $a_1 = a_2 = a$, $b_1 = 0 = b_2$.

c. The only input parameters for which an error in estimation is of

significance are D_0, R_x, and R_y; the other input parameters are assumed to be known exactly, thus having zero variances and covariances. Using this simple situation, the methodology used can be illustrated and more complex cases can be dealt with similarly. Under the stated assumptions one uses expression (112) with $a_1 = a_2 = a$ and $b_1 = b_2 = 0$. Substituting for the values of q_1^2 and q_2^2 as dictated by (79) and (80) in (112) results in

$$\hat{\Pi}_{kT}^{(2)} = R_x R_y \left(\frac{R_x^2}{2\,D_0} + \sigma_1^2 + \sigma_x^2\right)^{-\frac{1}{2}} \left(\frac{R_y^2}{2\,D_0} + \sigma_2^2 + \sigma_y^2\right)^{-\frac{1}{2}} \cdot \exp\left[-\frac{a^2}{2}\left(\frac{R_x^2}{2\,D_0} + \sigma_1^2 + \sigma_x^2\right)^{-1}\right]$$

$$- \frac{1}{4} R_x^2\, R_y^2 \left(\frac{R_x^2}{2\,D_0} + \sigma_1^2\right)^{-\frac{1}{2}} \left(\frac{R_x^2}{2\,D_0} + \sigma_1^2 + 2\sigma_x^2\right)^{-\frac{1}{2}}$$

$$\cdot \left(\frac{R_y^2}{2\,D_0} + \sigma_2^2\right)^{-\frac{1}{2}} \left(\frac{R_y^2}{2\,D_0} + \sigma_2^2 + 2\sigma_y^2\right)^{-\frac{1}{2}} \exp\left[-a^2\left(\frac{R_x^2}{2\,D_0} + \sigma_1^2\right)^{-1}\right] \qquad (124)$$

In general,

$$\hat{\Pi}_{kT}^{(2)} = \hat{\Pi}_{kT}^{(2)}\,(D_0,\ R_x,\ R_y) \qquad (125)$$

Let $\bar{D}_0$, $\bar{R}_x$, and $\bar{R}_y$ represent, respectively, the means of D_0, R_x, and R_y. Then, an approximate estimate of the mean of $\hat{\Pi}_{kT}^{(2)}$ is given by

$$E[\hat{\Pi}_{kT}^{(2)}] = \hat{\Pi}_{kT}^{(2)}\,(\bar{D}_0,\ \bar{R}_x,\ \bar{R}_y) \qquad (126)$$

The Taylor's series expansion will now be used to determine an approximate expression for $Var[\hat{\Pi}_{kT}^{(2)}]$. Expanding (125) as a Taylor's series about the mean of D_0, R_x, and R_y and retaining only first order terms yields

$$\hat{\Pi}_{kT}^{(2)}\,(D_0, R_x, R_y) = \hat{\Pi}_{kT}^{(2)}(\bar{D}_0, \bar{R}_x, \bar{R}_y) + (D - \bar{D}_0)\,\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial D_0}$$

$$+ (R_x - \bar{R}_x) \frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_x} + (R_y - \bar{R}_y) \frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_y} \tag{127}$$

where the partial derivatives are all computed at the point $(\bar{D}_0, \bar{R}_x, \bar{R}_y)$. Transposing $\hat{\Pi}_{kT}^{(2)}(\bar{D}_0, \bar{R}_x, \bar{R}_y)$ to the left-hand side, squaring and taking expectations on both sides of (127) results in the following expression for $\mathrm{Var}[\hat{\Pi}_{kT}^{(2)}]$.

$$\begin{aligned}
\mathrm{Var}[\hat{\Pi}_{kT}^{(2)}] = {} & \mathrm{Var}[D_0] \left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial D_0}\right)^2 + \mathrm{Var}[R_x] \left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_x}\right)^2 \\
& + \mathrm{Var}[R_y] \left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_y}\right)^2 + 2\mathrm{Cov}[D_0, R_x] \left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial D_0}\right)\left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_x}\right) \\
& + 2\,\mathrm{Cov}[D_0, R_y] \left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial D_0}\right)\left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_y}\right) \\
& + 2\,\mathrm{Cov}[R_x, R_y] \left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_x}\right)\left(\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_y}\right)
\end{aligned} \tag{128}$$

where again all the partial derivatives in (128) are computed at $(\bar{D}_0, \bar{R}_x, \bar{R}_y)$. It now remains to compute these partial derivatives from expression (124).

In expression (124) let

$$\begin{aligned}
A & = R_x R_y \left(\frac{R_x^2}{2\,D_0} + \sigma_1^2 + \sigma_x^2\right)^{-\frac{1}{2}} \left(\frac{R_y^2}{2\,D_0} + \sigma_2^2 + \sigma_y^2\right)^{-\frac{1}{2}} \exp\left[-\frac{a^2}{2}\left(\frac{R_x^2}{2\,D_0} + \sigma_1^2 + \sigma_x^2\right)^{-1}\right] \\
& = R_x R_y \,(q_1^2 + \sigma_x^2)^{-\frac{1}{2}} (q_2^2 + \sigma_y^2)^{-\frac{1}{2}} \exp\left[-\frac{a^2}{2}(q_1^2 + \sigma_x^2)^{-1}\right]
\end{aligned} \tag{129}$$

Let also

$$B = \frac{1}{4} R_x^2 R_y^2 \left(\frac{R_x^2}{2 D_0} + \sigma_1^2\right)^{-\frac{1}{2}} \left(\frac{R_x^2}{2 D_0} + \sigma_1^2 + 2\sigma_x^2\right)^{-\frac{1}{2}}$$

$$\cdot \left(\frac{R_y^2}{2 D_0} + \sigma_2^2\right)^{-\frac{1}{2}} \left(\frac{R_y^2}{2 D_0} + \sigma_2^2 + 2\sigma_y^2\right)^{-\frac{1}{2}} \exp\left[-a^2 \left(\frac{R_x^2}{2 D_0} + \sigma_1^2\right)^{-1}\right]$$

$$= \frac{1}{4} R_x^2 R_y^2 q_1^{-1} \left(q_1^2 + 2\sigma_x^2\right)^{-\frac{1}{2}} q_2^{-1} \left(q_2^2 + 2\sigma_y^2\right)^{-\frac{1}{2}} \exp(-a^2 q_1^{-2}) \qquad (130)$$

Thus,

$$\hat{\Pi}_{kT}^{(2)} = A - B \qquad (131)$$

and, in general,

$$\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial(\cdot)} = \frac{\partial A}{\partial(\cdot)} - \frac{\partial B}{\partial(\cdot)} \qquad (132)$$

Recall also from (79) and (80) that

$$q_1^2 = \frac{R_x^2}{2 D_0} + \sigma_1^2 \qquad (133)$$

$$q_2^2 = \frac{R_y^2}{2 D_0} + \sigma_2^2 \qquad (134)$$

The following is obtained

$$\frac{\partial A}{\partial D_0} = -\frac{A}{4 D_0^2} \left[R_x^2 (q_1^2 + \sigma_x^2)^{-1} + R_y^2 (q_2^2 + \sigma_y^2)^{-1} - a^2 R_x^2 (q_1^2 + \sigma_x^2)^{-2}\right] \qquad (135)$$

$$\frac{\partial B}{\partial D_0} = \frac{B}{4 D_0^2} \left[R_x^2 (q_1^2)^{-1} + R_x^2 (q_1^2 + 2\sigma_x^2)^{-1} + R_y^2 (q_2^2)^{-1} + R_y^2 (q_2^2 + 2\sigma_y^2)^{-1}\right.$$

$$\left. - 2a^2 R_x^2 (q_1^2)^{-2}\right] \qquad (136)$$

$$\frac{\partial A}{\partial R_x} = A\left[\frac{1}{R_x} - \frac{R_x}{2\,D_0}(q_1^2 + \sigma_x^2)^{-1} + \frac{a^2\,R_x}{2\,D_0}(q_1^2 + \sigma_x^2)^{-2}\right] \tag{137}$$

$$\frac{\partial B}{\partial R_x} = B\left[\frac{2}{R_x} - \frac{R_x}{2\,D_0}(q_1^2)^{-1} - \frac{R_x}{2\,D_0}(q_1^2 + 2\sigma_x^2)^{-1} + \frac{a^2\,R_x}{D_0}(q_1^2)^{-2}\right] \tag{138}$$

$$\frac{\partial A}{\partial R_y} = A\left[\frac{1}{R_y} - \frac{R_y}{2\,D_0}({q^2}_2 + \sigma_y^2)^{-1}\right] \tag{139}$$

$$\frac{\partial B}{\partial R_y} = B\left[\frac{2}{R_y} - \frac{R_y}{2\,D_0}(q_2^2)^{-1} - \frac{R_y}{2\,D_0}(q_2^2 + 2\sigma_y^2)^{-1}\right] \tag{140}$$

5. Example

The following data are provided

Weapon: 5EAKL; Target: 3172; Impact angle: 75°; Impact Velocity: 900 ft/sec

D_0 = .594,85; R_x = 59.21 ft; R_y = 122.92 ft.

Var[D_0] = .000,29; Var[R_x] = 2.120,4; Var[R_y] = 8.268,2

Cov[D_0,R_x] = .000,61; Cov[$D_0 R_y$] = -.000,43; Cov[$R_x R_y$] = -1.474,43

σ_1 = 30 ft; σ_2 = 20 ft; σ_x = 150 ft; σ_y = 100 ft

It is required to determine the following:

- The optimum value of a which determines the stick pattern.
- The maximum probability of kill.
- The error on the probability of kill given the variance and covariance on the impact parameters D_0, R_x and R_y.

Consider a stick pattern where $a_1 = -a_2$ and $b_1 = b_2 = 0$

Solution

The following quantities of interest are to be computed:

1. q_1^2 and q_1 from (79)

2. q_2^2 and q_2 from (80)

3. C_1 from (113)

4. C_2 from (114)

5. The optimal value of a from (119)

6. The maximum value of $\hat{\Pi}_{kT}^{(2)}$ from (116)

7. The value of A from (129)

8. The value of B from (130)

9. (check) The value of $\hat{\Pi}_{kT}^{(2)}$ from (131)

10. The values of $\frac{\partial A}{\partial D_0}$, $\frac{\partial B}{\partial D_0}$, $\frac{\partial A}{\partial R_x}$, $\frac{\partial B}{\partial R_x}$, $\frac{\partial A}{\partial R_y}$, and $\frac{\partial B}{\partial R_y}$ from (135) to (140)

11. The values of $\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial (\cdot)}$ using (132)

12. $\mathrm{var}[\hat{\Pi}_{kT}^{(2)}]$ from (128)

1. Compute q_1 from (79)

$$q_1^2 = \frac{R_x^2}{2\ D_0} + \sigma_1^2$$

$$= \frac{(59.21)^2}{2(.594,85)} + (30)^2$$

$$= 3,846.813,6$$

$$q_1 = 62.022,69$$

2. Compute q_2 from (80)

$$q_2^2 = \frac{R_y^2}{2\ D_0} + \sigma_2^2$$

$$= \frac{(122.92)^2}{2(.594,85)} + (20)^2$$

$$= 13,100.114,7$$

$$q_2 = 114.455,73.$$

3. Compute C_1 from (113)

$$C_1 = \frac{R_x R_y}{2} \frac{1}{\sqrt{q_1^2 + \sigma_x^2}} \cdot \frac{1}{\sqrt{q_2^2 + \sigma_y^2}}$$

$$C_1 = \frac{(59.21)(122.92)}{2} \cdot \frac{1}{\sqrt{(3,846.813,6) + (150)^2}}$$

$$\cdot \frac{1}{\sqrt{(13,100.114,7) + (100)^2}}$$

$$= \frac{(59.21)(122.92)}{(2)(162.317,0)(151.987,2)} = .147,508,3$$

4. Compute C_2 from (114)

$$C_2 = \frac{R_x^2}{2q_1 \sqrt{2\sigma_x^2 + q_1^2}} \cdot \frac{R_y^2}{2q_2 \sqrt{2\sigma_y^2 + q_2^2}}$$

$$= \frac{(59.21)^2}{(2)(62.022,69) \sqrt{(2)(150)^2 + 3,846.813,6}}$$

$$\cdot \frac{(122.92)^2}{(2)(114.455,73) \sqrt{(2)(100)^2 + 13,100.114,7}}$$

$$= (.127,876,7)(.362,796,3) = .046,393,2$$

5. Compute the optimal a from (119)

$$a^2 = \frac{\ln\left(\frac{C_2}{C_1} \; \frac{q_1^2 + \sigma_x^2}{q_1^2}\right)}{\frac{1}{q_1^2} - \frac{1}{2}\,\frac{1}{q_1^2 + \sigma_x^2}}$$

$$= \frac{\ln\left(\frac{.046,393,2}{.147,508,3} \; \frac{3,846.813,6 + (150)^2}{3,846.813,6}\right)}{\frac{1}{3,846.813,6} - \frac{1}{2}\,\frac{1}{3,846.813,6 + (150)^2}}$$

$$= \frac{.767,370,6}{.000,259,955,4 - .000,018,977,5}$$

$$= 3,184.403,7 \text{ ft}^2$$

$$a = 56.430,5 \text{ ft.}$$

6. Compute $\hat{\pi}_{kT}^{(2)}$ from (116)

$$\hat{\pi}_{kT}^{(2)} = 2\, C_1 \exp\left(-\frac{1}{2}\,\frac{a^2}{q_1^2 + \sigma_x^2}\right) - C_2 \exp\left(-\frac{a^2}{q_1^2}\right)$$

$$= (2)(.147,508,3) \exp\left[-\frac{1}{2}\,\frac{3,184.403,7}{3,846.813,6 + (150)^2}\right]$$

$$- (.046,393,2) \exp\left(-\frac{1}{2}\,\frac{3,184.403,7}{3,846.813,6}\right)$$

$$= .277,716,053,2 - .020,274,2160$$

$$= .257,44.$$

7. Compute A from (129)

$$= R_x R_y \left(q_1^2 + \sigma_x^2\right)^{-\frac{1}{2}} \left(q_2^2 + \sigma_y^2\right)^{-\frac{1}{2}} \exp\left[-\frac{a^2}{2}\left(q_1^2 + \sigma_x^2\right)^{-1}\right]$$

$$= (59.21)(122.92)\left[3{,}846.813{,}6 + (150)^2\right]^{-\frac{1}{2}}$$

$$\left[13{,}100.114{,}7 + (100)^2\right]^{-\frac{1}{2}} \exp\left[-\frac{3{,}184.403{,}7}{2}\left[3{,}846.813{,}6 + (150)^2\right]^{-1}\right]$$

$$A = (59.21)(122.92)(.006{,}160{,}783{,}5)(.006{,}579{,}500{,}6)\ \exp\ (-.060{,}432{,}425{,}5)$$

$$= .277{,}716{,}077.$$

8. Compute B from (130)

$$B = \frac{1}{4} R_x^2\ R_y^2\ q_1^{-1}\left(q_1^2 + 2\sigma_x^2\right)^{-\frac{1}{2}} q_2^{-1}\left(q_2^2 + 2\sigma_y^2\right)^{-\frac{1}{2}} \exp\left(-a^2\ q_1^{-2}\right)$$

$$= \frac{1}{4}(59.21)^2\ (122.92)^2\ (62.022{,}69)^{-1}\left[3{,}846.813{,}6 + (2)(150)^2\right]^{-\frac{1}{2}}$$

$$\cdot\ (114.455{,}73)^{-1}\left[13{,}100.114{,}7 + (2)(100)^2\right]^{-\frac{1}{2}} \exp\left(-\frac{3{,}184.403{,}7}{3{,}846.813{,}6}\right)$$

$$= \frac{1}{4}(59.21)^2(122.92)^2(.016{,}123{,}131{,}7)(.004{,}524{,}617{,}8)$$

$$(.008{,}737{,}002{,}5)(.005{,}496{,}487{,}6)(.437{,}008{,}355{,}5)$$

$$= .020{,}274{,}212{,}2.$$

9. Check the value of $\hat{\Pi}_{kT}^{(2)}$ from (131)

$$\hat{\Pi}_{kT}^{(2)} = A - B$$

$$= .277{,}160{,}077 - .020{,}274{,}212$$

$$= .257{,}44$$

10. Compute the value of the partial derivatives from (135) to (140)

a. $\frac{\partial A}{\partial D_0}$ from (135)

$$\frac{\partial A}{\partial D_0} = \frac{A}{4\ D_0^2}\left[R_x^2\ (q_1^2 + \sigma_x^2)^{-1} + R_y^2\ (q_2^2 + \sigma_y^2)^{-1} - a^2\ R_x^2\ (q_1^2 + \sigma_x^2)^{-2}\right]$$

$$= \frac{.277{,}716{,}077}{4(.594.85)^2}\ \{(59.21)^2\ [3{,}846.813{,}6 + (150)^2]^{-1}$$

$$+ (122.92)^2\ [13{,}100.114{,}7 + (100)^2]^{-1}$$

$$- 3{,}184.403{,}7\ (59.21)^2\ [3{,}846.813{,}6 + (150)^2]^{-2}\}$$

$$= \frac{.277{,}716{,}077}{(4)(.594{,}85)^2}\ (.133{,}064{,}443{,}9 + .654{,}080{,}146{,}2 - .016{,}082{,}814{,}2)$$

$$= .151{,}291{,}759{,}2.$$

b. $\frac{\partial B}{\partial D_0}$ from (136)

$$\frac{\partial B}{\partial D_0} = \frac{B}{4\ D_0^2}\ \{R_x^2\ (q_1^2)^{-1} + R_x^2\ (q_1^2 + 2\sigma_x^2)^{-1} + R_y^2\ (q_2^2)^{-1}$$

$$+ R_y^2\ (q_2^2 + 2\sigma_y^2)^{-1} - 2a^2\ R_x^2\ (q_1^2)^{-2}]$$

$$= \frac{.020{,}274{,}212{,}2}{(4)(.594{,}85)^2}\ \{(59.21)^2\ (3{,}846.813{,}6)^{-1}$$

$$+ (59.21)^2\ [3{,}846.813{,}6 + 2(150)^2]^{-1}$$

$$+ (122.92)^2\ (13{,}100.114{,}7)^{-1}$$

$$+ (122.92)^2 \left[13{,}100.114{,}7 + 2(100)^2\right]^{-1}$$

$$- (2)(3{,}184.403{,}7)(59.21)^2 \ (3{,}846.813{,}6)^{-2}\}$$

$$\frac{\partial B}{\partial D_0} = \frac{.020{,}274{,}212{,}2}{4(.594{,}85)^2} (.911{,}357{,}3 + .071{,}771{,}807{,}4$$

$$+ 1.153{,}373{,}596 + .456{,}473{,}536 - 1.508{,}849{,}6)$$

$$= .015{,}529{,}209.$$

c. $\frac{\partial A}{\partial R_x}$ from (137)

$$\frac{\partial A}{\partial R_x} = A \left[\frac{1}{R_x} - \frac{R_x}{2D_0} (q_1^2 + \sigma_x^2)^{-1} + \frac{a^2 R_x}{2D_0} (q_1^2 + \sigma_x^2)^{-2} \right]$$

$$= (.277{,}716{,}077) \left\{ \frac{1}{59.21} - \frac{59.21}{(2)(.594{,}85)} \left[3{,}846.813{,}6 + (150)^2\right]^{-1} \right.$$

$$\left. + \frac{(3{,}184.403{,}7)(59.21)}{(2)(.594{,}85)} \left[3{,}846.813{,}6 + (150)^2\right]^{-2} \right\}$$

$$= (.277{,}716{,}077)(.016{,}889{,}039 - .001{,}888{,}989{,}3 + .000{,}228{,}312{,}4)$$

$$= .004{,}229{,}161.$$

d. $\frac{\partial B}{\partial R_x}$ from (138)

$$\frac{\partial B}{\partial R_x} = B \left[\frac{2}{R_x} - \frac{R_x}{2 D_0} (q_1^2)^{-1} - \frac{R_x}{2 D_0} (q_1^2 + 2\sigma_x^2)^{-1} + \frac{a^2 R_x}{D_0} (q_1^2)^{-2} \right]$$

$$= .020{,}274{,}212{,}2 \left\{ \frac{2}{59.21} - \frac{59.12}{(2)(.594{,}85)} (3{,}846.813{,}6)^{-1} \right.$$

$$- \frac{59.21}{(2)(.594{,}85)} \left[3{,}846.813{,}6 + (2)(150)^2 \right]^{-1}$$

$$+ \frac{(3{,}184.403{,}7)(59.21)}{(.594{,}85)} (3{,}846{,}813{,}6)^{-2} \}$$

$$\frac{\partial B}{\partial R_x} = .020{,}274{,}212{,}2 \; (.033{,}778{,}078 - .012{,}937{,}681{,}5$$

$$- .001{,}018{,}876{,}1 + .021{,}419{,}702{,}2)$$

$$= .000{,}836.133{,}3.$$

e. $\frac{\partial A}{\partial R_y}$ from (139)

$$\frac{\partial A}{\partial R_y} = A \left[\frac{1}{R_y} - \frac{R_y}{2\,D_0} (q_2^2 + \sigma_y^2)^{-1} \right]$$

$$= .277{,}716{,}077 \left\{ \frac{1}{122.92} - \frac{122.92}{(2)(.594{,}85)} \left[13{,}100.114{,}7 + (100)^2 \right] \right\}$$

$$= .277{,}716{,}077 \; (.008{,}135{,}372{,}6 - .004{,}472{,}7122)$$

$$= .001{,}017{,}179{,}7.$$

f. $\frac{\partial B}{\partial R_y}$ from (140)

$$\frac{\partial B}{\partial R_y} = B \left[\frac{2}{R_y} - \frac{R_y}{2\,D_0} (q_2^2)^{-1} - \frac{R_y}{2\,D_0} (q_2^2 + 2\sigma_y^2)^{-1} \right]$$

$$= .020{,}274{,}212{,}2 \left\{ \frac{2}{122.92} - \frac{122.92}{(2)(.594{,}85)} (13{,}100.114{,}7)^{-1} \right.$$

$$\left. - \frac{122.92}{(2)(.594{,}85)} \left[13{,}100.114{,}7 + (2)(100)^2 \right]^{-1} \right.$$

$$= .020{,}274{,}212{,}2 \; (.016{,}270.745{,}2 - .007{,}886{,}966{,}4 - .003{,}121{,}444{,}3)$$

$$= .000{,}106{,}689{,}7.$$

11. Compute $\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial (.)}$ from (132)

a. $$\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial D_0} = \frac{\partial A}{\partial D_0} - \frac{\partial B}{\partial D_0}$$

$$= .151,291,759,2 - .015,529,209$$

$$= .135,762,550,2$$

b. $$\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_x} = \frac{\partial A}{\partial R_x} - \frac{\partial B}{\partial R_x}$$

$$= .004,229,161 - .000,836,133,3$$

$$= .003,393,027,7$$

c. $$\frac{\partial \hat{\Pi}_{kT}^{(2)}}{\partial R_y} = \frac{\partial A}{\partial R_y} - \frac{\partial B}{\partial R_y}$$

$$= .001,017,179,7 - .000,106,689,7$$

$$= .000,910,49$$

12. Compute the value of $\mathrm{Var}[\hat{\Pi}_{kT}^{(2)}]$ from (128)

$$\mathrm{Var}[\hat{\Pi}_{kT}^{(2)}] = (.000,29)(.135,762,550,2)^2$$

$$+ (2.130,4)(.003,393,027,7)^2$$

$$+ (8.268,2)(.000,910,49)^2$$

$$+ (2)(.000,61)(.135,762,550,2)(.003,393,027,7)$$

$$+ (2)(-.000,43)(.135,762,550,2)(.000,910,49)$$

$$+ (2)(-1.474,3)(.003,393,027,7(.000,910,49)$$

$$
\begin{aligned}
\mathrm{Var}[\hat{\pi}_{kT}^{(2)}] &= .000{,}005{,}345{,}1 \\
&+ .000{,}024{,}526{,}5 \\
&+ .000{,}006{,}854{,}3 \\
&+ .000{,}000{,}562{,}0 \\
&- .000{,}000{,}106{,}3 \\
&- .000{,}009{,}109{,}2 \\
&= .000{,}028{,}072{,}4.
\end{aligned}
$$

Note that $\mathrm{Var}[R_x]$ contributes 87.36% of the total variance. The standard error on $\hat{\pi}_{kT}^{(2)}$ is

$$\sigma_{\hat{\pi}_{kT}^{(2)}} = \sqrt{.000{,}028{,}072{,}4} = .005{,}3$$

A two standard error confidence interval on $\hat{\pi}_{kT}^{(2)}$ is

$$\hat{\pi}_{kT}^{(2)} \pm 2\,\sigma_{\hat{\pi}_{kT}^{(2)}} = .257{,}4 \pm .010{,}6.$$

SECTION IX

CONCLUSIONS AND RECOMMENDATIONS

A theoretical model is formulated to provide an expression for the probability of kill of a fragment sensitive target when hit by a stick of weapons. Each weapon is assumed to be subject to ballistic errors, and the stick pattern itself is assumed to be subject to an aiming error.

A detailed analysis is provided when the stick consists of two weapons. This analysis includes the following:

a. The determination of the optimum stick pattern.

b. The evaluation of the probability of kill of a point target.

c. The determination of the variance of the probability of kill given that the input parameters are subject to estimation error.

Although the mathematical analysis becomes quite cumbersome when the number of weapons in the stick pattern becomes greater than two, nevertheless, it is recommended that alternative methods, such as numerical analysis and/or recursive schemes, be investigated to provide computationally valid ways of analyzing the system for an arbitrary number n of weapons. Specifically, the analysis would result in the following:

a. The determination of the optimum stick pattern.

b. The numerical evaluation of the probability of kill.

c. The determination of a numerically computable expression for $Var[P_k]$, given the variance-covariance matrix of the seven input parameters.

It is also recommended that a similar analysis be performed for the case involving stick delivery of multiple weapons whose main effect is blast.

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APPENDIX A
VALIDATION OF AN EXPRESSION

The purpose of this appendix is to establish the validity of expression (11).

Define as (u,v) the coordinates of the point target. For the ith weapon, $i=1,2,\ldots,n$, suppose that it is aimed at (ξ_i, η_i) so that (ξ_i, η_i) is the MPI. Let the ith weapon subject to ballistic error impact at (x_i, y_i). The ballistic error in the range direction is $(x_i-\xi_i)$ and in the deflection direction is $(y_i-\eta_i)$ · $(x_i-\xi_i)$ and $(y_i-\eta_i)$ are assumed to be independently distributed each having a Gaussian distribution with zero mean and respective standard deviation σ_{1i} and σ_{2i}. $(x_i-\xi_i)$ and $(y_i-\eta_i)$ are also assumed to be independently distributed between weapons. If $f_{1i}(\cdot)$ and $f_{2i}(\cdot)$ represent the respective probability density functions of $(x_i-\xi_i)$ and $(y_i-\eta_i)$, one has

$$f_{1i}\,(x_i-\xi_i) = \frac{1}{\sigma_{1i}\sqrt{2\pi}} \exp\Big[-\frac{(x_i-\xi_i)^2}{2\sigma_{1i}^2}\Big] \qquad i=1,2,\ldots,n \qquad (A-1)$$

$$f_{2i}\,(y_i-\eta_i) = \frac{1}{\sigma_{2i}\sqrt{2\pi}} \exp\Big[-\frac{(y_i-\eta_i)^2}{2\sigma_{2i}^2}\Big] \qquad i=1,2,\ldots,n \qquad (A-2)$$

Now, since all weapons are identical, the probability of kill at (u,v) for weapon i, given that it impacts at (x_i, y_i), is given by the three-parameter Carleton damage function

$$D(u-x_i,\ v-y_i) = D_0 \exp\Big\{-D_0\Big[\Big(\frac{u-x_i}{R_x}\Big)^2 + \Big(\frac{v-y_i}{R_y}\Big)^2\Big]\Big\} \qquad (A-3)$$

If weapon i had acted individually, without the contribution of the other weapons, the resultant probability of kill of the point target (u,v) would be given by (10) or

$$P_{k_i}(u-\xi_i,\ v-\eta_i) = \frac{R_x R_y}{2q_{1i}\ q_{2i}} \exp\{-\frac{1}{2}[(\frac{u-\xi_i}{q_{1i}})^2 + (\frac{v-\eta_i}{q_{2i}})^2\} \quad \text{(A-4)}$$

where
$$q_{1i} = \frac{R_x^2}{2\ D_0} + \sigma_{1i}^2 \quad \text{(A-5)}$$

$$q_{2i} = \frac{R_y^2}{2\ D_0} + \sigma_{2i}^2 \quad \text{(A-6)}$$

It is required to show that the kill contribution of each weapon in the presence of ballistic error is independent of the kill contribution of any other weapon so that the net probability of kill of the point target at (u,v) is given by the well known formula

$$\hat{P}_k(u,v) = 1 - \prod_{i=1}^{n} [1 - P_{k_i}(u-\xi_i,\ v-\eta_i)] \quad \text{(A-7)}$$

The result will be shown for the case of n=2 weapons. The method can easily be extended to an arbitrary number of weapons. Now

Probability of kill at (u,v) = $P_k^{(2)}$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$ [Probability of kill at (u,v)|weapon 1 impacts at (x_1,y_1) and weapon 2 impacts at (x_2,y_2)] [Probability weapon 1 impacts between (x_1,y_1) and $(x_1 + dx_1,\ y_1 + dy_1)$ and weapon 2 impacts between (x_2,y_2) and $(x_2 + dx_2, y_2 + dy_2)$] (A-8)

But,

[Probability of kill at $(u,v)|$weapon 1 impacts at (x_1,y_1) and weapon 2 impacts at (x_2,y_2)] =

$$1 - [1 - D(u-x_1,\ v-y_1)][1-D(u-x_2,\ v-y_2)] \tag{A-9}$$

Using (A-1), (A-2), and (A-9) in (A-8) results in

$$\hat{P}_k^{(2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{1 - [1- D(u-x_1,\ v-y_1)][1-D(u-x_2,\ v-y_2)]$$

$$f_{11}(x_1-\xi_1)\ f_{21}(y_1-\eta_1)\ f_{12}(x_2-\xi_2)\ f_{22}(y_2-\eta_2)\ dx_1\ dy_1\ dx_2\ dy_2 \tag{A-10}$$

Reducing expression (A-9) and using it in (A-10) results in

$$\hat{P}_k^{(2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(u-x_1,v-y_1)\ f_{11}(x_1-\xi_1)\ f_{21}(y_1-\eta_1)\ dx_1dy_1$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(u-x_2,v-y_2)\ f_{12}(x_2-\xi_2)\ f_{22}(y_2-\eta_2)\ dx_2dy_2$$

$$- [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(u-x_1,v-y_1)\ f_{11}(x_1-\xi_1)\ f_{21}(y_1-\eta_1)\ dx_1dy_1]$$

$$[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(u-x_2,v-y_2)\ f_{12}(x_2-\xi_2)\ f_{22}(y_2-\eta_2)\ dx_2dy_2]$$

The integrals in (A-11) can be reduced to form (6) whose explicit evaluation is of the form (7). Thus, using (A-4)

$$\hat{P}_k^{(2)} = P_{k1}(u-\xi_1,v-\eta_1) + P_{k2}(u-\xi_2,v-\eta_2) - P_{k1}(u-\xi_1,v-\eta_1) \cdot P_{k2}(u-\xi_2,v-\eta_2)$$

$$= 1 - [1 - P_{k1}(u-\xi_1,\ v-\eta_1)]\ [1 - P_{k2}(u-\xi_2,\ v-\eta_2)] \tag{A-12}$$

If the ballistic errors have the same standard deviations, then

$$\sigma_{1i}^2 = \sigma_1^2 \qquad \text{for all } i$$

$$\sigma_{2i}^2 = \sigma_2^2 \qquad \text{for all } i$$

$$q_{1i} = q_1 \qquad \text{for all } i$$

$$q_{2i} = q_2 \qquad \text{for all } i$$

and $P_{ki}(u-\xi_i, v-\eta_i) = P_k(u-\xi_i, v-\eta_i)$

APPENDIX B

EVALUATION OF AN INTEGRAL

It is required to evaluate the following integral

$$I_n(\alpha_1,\alpha_2,\ldots,\alpha_n;\ \beta_1,\beta_2,\ldots,\beta_n) = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\frac{(x+\alpha_i)^2}{\beta_i^2}\right] dx \tag{B-1}$$

The well-known formula

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy = \sqrt{2\pi} \tag{B-2}$$

will be used in the sequel.

The problem will be solved for n=1 and n=2 and a general expression for (B-1) will be inferred.

Case when n=1

The purpose here is not in obtaining a final reduced answer (which is $\beta_1\sqrt{2\pi}$), but rather to obtain an expression whose form can be generalized for any n. From (B-1)

$$I_1(\alpha_1,\beta_1) = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\frac{(x+\alpha_1)^2}{\beta_1^2}\right] dx \tag{B-3}$$

Now

$$\frac{(x+\alpha_1)^2}{\beta_1^2} = \frac{1}{\beta_1^2} \left(x^2 + 2 \frac{\frac{\alpha_1}{\beta_1^2}}{\frac{1}{\beta_1^2}} x + \frac{\frac{\alpha_1^2}{\beta_1^2}}{\frac{1}{\beta_1^2}}\right) \tag{B-4}$$

Let

$$\gamma_1 = \frac{1}{\beta_1^2} \tag{B-5}$$

$$\delta_1 = \frac{\alpha_1^2}{\beta_1^2} \tag{B-6}$$

$$\varepsilon_1 = \frac{\alpha_1}{\beta_1^2} \tag{B-7}$$

Using (B-5), (B-6), and (B-7) in (B-4) yields

$$\frac{(x+\alpha_1)^2}{\beta_1^2} = \gamma_1 \left(x^2 + 2 \frac{\varepsilon_1}{\gamma_1} x + \frac{\delta_1}{\gamma_1}\right) \tag{B-8}$$

$$= \gamma_1 \left[\left(x + \frac{\varepsilon_1}{\gamma_1}\right)^2 + \frac{\delta_1}{\gamma_1} - \frac{\varepsilon_1^2}{\gamma_1^2}\right] \tag{B-9}$$

$$= \gamma_1 \left(x + \frac{\varepsilon_1}{\gamma_1}\right)^2 + \delta_1 - \frac{\varepsilon_1^2}{\gamma_1} \tag{B-10}$$

Substituting (B-10) in (B-3) results in

$$I_1(\alpha_1,\beta_1) = \int_{-\infty}^{\infty} \exp\{- \tfrac{1}{2} [\gamma_1(x + \frac{\varepsilon_1}{\gamma_1})^2 + \delta_1 - \frac{\varepsilon_1^2}{\gamma_1}]\} \, dx$$

$$= \int_{-\infty}^{\infty} \exp[-\frac{1}{2}\gamma_1 (x + \frac{\varepsilon_1}{\gamma_1})^2]\, dx.\ \exp[-\frac{1}{2}(\delta_1 - \frac{\varepsilon_1^2}{\gamma_1})] \tag{B-11}$$

In the integral of (B-11), let

$$\sqrt{\gamma_1}\,(x + \frac{\varepsilon_1}{\gamma_1}) = y \tag{B-12}$$

then

$$dx = \frac{1}{\sqrt{\gamma_1}}\, dy \tag{B-13}$$

and (B-11) becomes

$$I_1(\alpha_1,\beta_1) = \frac{1}{\sqrt{\gamma_1}} \int_{-\infty}^{\infty} \exp(-\frac{y^2}{2})\, dy \cdot \exp[-\frac{1}{2}(\delta_1 - \frac{\varepsilon_1^2}{\gamma_1})]$$

$$= \sqrt{\frac{2\pi}{\gamma_1}}\ \exp[-\frac{1}{2}(\delta_1 - \frac{\varepsilon_1^2}{\gamma_1})] \tag{B-14}$$

γ_1, δ_1, and ε_1 are as defined in (B-5), (B-6), and (B-7), respectively.

<u>Case when n=2</u>

$$I_2(\alpha_1,\alpha_2;\beta_1,\beta_2) = \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}[\frac{(x+\alpha_1)^2}{\beta_1^2} + \frac{(x+\alpha_2)^2}{\beta_2^2}]\}\, dx \tag{B-15}$$

Now

$$\frac{(x+\alpha_1)^2}{\beta_1^2} + \frac{(x+\alpha_2)^2}{\beta_2^2} = \frac{x^2}{\beta_1^2} + 2\frac{\alpha_1}{\beta_1^2}x + \frac{\alpha_1^2}{\beta_1^2} + \frac{x^2}{\beta_2^2} + 2\frac{\alpha_2}{\beta_2^2}x + \frac{\alpha_2^2}{\beta_2^2}$$

$$= (\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2})\, x^2 + 2\,(\frac{\alpha_1}{\beta_1^2} + \frac{\alpha_1}{\beta_2^2})\, x + (\frac{\alpha_1^2}{\beta_1^2} + \frac{\alpha_2^2}{\beta_2^2}) \tag{B-16}$$

Let

$$\gamma_2 = \frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} \qquad \text{(B-17)}$$

$$\delta_2 = \frac{\alpha_1^2}{\beta_1^2} + \frac{\alpha_2^2}{\beta_2^2} \qquad \text{(B-18)}$$

$$\varepsilon_2 = \frac{\alpha_1}{\beta_1^2} + \frac{\alpha_2}{\beta_2^2} \qquad \text{(B-19)}$$

Using (B-17), (B-18), and (B-19) in (B-16) yields

$$\frac{(x+\alpha_1)^2}{\beta_1^2} + \frac{(x+\alpha_2)^2}{\beta_2^2} = \gamma_2 \left(x^2 + 2\,\frac{\varepsilon_2}{\gamma_2} + \frac{\delta_2}{\gamma_2}\right) \qquad \text{(B-20)}$$

$$= \gamma_2 \left[\left(x + \frac{\varepsilon_2}{\gamma_2}\right)^2 + \frac{\delta_2}{\gamma_2} - \frac{\varepsilon_2^2}{\gamma_2^2}\right] \qquad \text{(B-21)}$$

$$= \gamma_2 \left(x + \frac{\varepsilon_2}{\gamma_2}\right)^2 + \delta_2 - \frac{\varepsilon_2^2}{\gamma_2} \qquad \text{(B-22)}$$

Substituting (B-22) in (B-15) results in

$$I_2(\alpha_1, \alpha_2; \beta_1, \beta_2) =$$

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\gamma_2 \left(x + \frac{\varepsilon_2}{\gamma_2}\right)^2 + \delta_2 - \frac{\varepsilon_2^2}{\gamma_2}\right]\right\} \qquad \text{(B-23)}$$

It immediately follows that

$$I_2(\alpha_1,\alpha_2; \beta_1,\beta_2) = \sqrt{\frac{2\pi}{\gamma_2}}\; \exp\left[-\frac{1}{2}\left(\delta_2 - \frac{\varepsilon_2^2}{\gamma_2}\right)\right] \qquad \text{(B-24)}$$

γ_2, δ_2 and ε_2 are as defined in (B-17), (B-18) and (B-19) respectively.

General Case

Let

$$\gamma_n = \sum_{i=1}^{n} \frac{1}{\beta_i^2} \tag{B-25}$$

$$\delta_n = \sum_{i=1}^{n} \frac{\alpha_i^2}{\beta_i^2} \tag{B-26}$$

$$\varepsilon_n = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i^2} \tag{B-27}$$

then

$$I_n(\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n) = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\frac{(x+\alpha_i)^2}{\beta_i^2}\right] dx$$

$$= \sqrt{\frac{2\pi}{\gamma_n}} \exp\left[-\frac{1}{2}\left(\delta_n - \frac{\varepsilon_n^2}{\gamma_n}\right)\right] \tag{B-28}$$

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